THE TEACHING OF ARITHMETIC

LENNES

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THE TEACHING OF ARITHMETIC

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THE TEACHING OF ARITHMETIC

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INTRODUCTION

In discussing the teaching of any subject one can not avoid the consideration of certain more general problems in education. Courses of study, topics to be included and excluded, the purpose and general quality of classroom teaching — all these depend largely upon fundamental views about the ends to be attained and the possible means of attaining them. In PART ONE of this book certain material bearing on these points has been collected for the convenience of the reader and an attempt made to evaluate it in so far as it has a bearing on the teaching of arithmetic.

The following questions are considered in the four chapters which constitute PART ONE:

1. What is the modern doctrine in regard to formal discipline and what is the character of the criticism and experimentation which, as is quite generally supposed, have destroyed or at least fundamentally modified the older doctrine? Precisely what kinds of effects of learning have been shown not to "carry over" into fields not specifically covered in the educational process, and what effects, if any, of this process are we still free to believe do carry over? What elements in our teaching should be stressed to secure the maximum of inclination and ability to cope intelligently with the strange situations that arise constantly in that practical life for which education in the schools is regarded as a preparation?

2. How does learning in school differ from learning in practical life outside the school? Are the methods of learning in the schools essentially superior to those of the outside world or should we seek to make them more "natural" and "practical" by

13481

adopting, as nearly as may be, the methods by which the adult learns in his workaday life? Is the "topical method" of teaching and learning superior or inferior to some other method such as the "project method" in which a natural life situation forms the center of interest and attack?

3. What are the native interests of the young child and how are these related to the work in the elementary grades, especially in arithmetic? To what extent is information about numbers necessary for the living of a full and uncramped child life? How can the number work of the elementary grades be so motivated that the child will attack it with a vehement purpose to learn and continue his work with zest and pleasure?

4. What new interests arise in the child during the years of his life in the grades, which ones of his early interests continue in full force, and which ones decrease in strength? What is the character of the interest which the normal child in the upper grades takes in the practical world of the adult? What elements of motivation used in the elementary grades should be discontinued and what new ones should be introduced?

These questions are pertinent in considering the teaching of any school subject whatever though they may seem to find a more appropriate place in a general treatise on education. They are, however, so closely bound up with the teaching of arithmetic and are so fundamental, especially from the author's point of view, that a discussion of them in this book seems necessary. Moreover, their magnitude is so great that incidental discussion in connection with the treatment of the teaching of the various topics in arithmetic would be inadequate and piecemeal and would lead to unfortunate and distracting diversions. For these reasons PART ONE of this book is devoted entirely to the consideration of these questions.

When, for practical purposes at least, more or less tentative conclusions have been reached on these questions, the practical teacher and school administrator are beset with a bewildering array of questions on the teaching of arithmetic itself. In PART TWO such questions are considered, in so far as the writer has been able to do so, in the light of the best information now available. These questions may be grouped under general heads among which the following are important:

1. What are the main purposes to be achieved in a school course in arithmetic?
2. How should the fundamental combinations be learned and habituated?
3. How should the fundamental operations be learned?
4. How should the solution of problems be developed?
5. How should the sources of problems be studied so as to make the subject appear as a living and working element in the practical world? How may arithmetic be socialized?
6. What are the needs for arithmetic in the home and on the farm, and how may these needs be met?
7. How may the teacher test the results of her teaching?
8. How may the principles of arithmetic be grouped about a few far-reaching ideas which will simplify learning and make retention more secure?

The list might be greatly extended. There are incidental but highly important topics, such as the saving and investing of money, the keeping of records of expenditures which will enable a family to know, for instance, what part of its total expenditures goes for a car, a servant, or any other given purpose.

Finally, it is realized more and more that the problem of the schools is not only to teach the processes of arithmetic but also to point out in a vital way the many important uses to which it may be put with real profit in terms of economic welfare. There is no more fundamental mistake than to suppose that the average person realizes the existence of the problems which he should solve. It is no exaggeration to say that every adult, no matter what his occupation, could spend some time each week keeping records of one kind and another and in doing other work involving the use of arithmetic which would yield a return in dollars and cents per hour that would be greater than that which he receives for any other work that he does. To bring this fact home to pupils in a vital way is one of the chief purposes of a course in arithmetic. Numerous problems of this kind are pointed out in PART TWO of this book, and others will be found by the teacher and the pupils when their attention is definitely directed to the problem of determining the maximum of profitable uses to which arithmetic may be put.

TABLE OF CONTENTS

PART I

GENERAL PROBLEMS BEARING ON THE TEACHING OF ARITHMETIC

CHAPTER PAGE

I. FORMAL DISCIPLINE 3

II. METHODS OF LEARNING AND TEACHING 69

III. MOTIVATION IN THE EARLY GRADES 119

IV. MOTIVATION IN THE GRAMMAR GRADES 155

PART II

SPECIAL PROBLEMS IN THE TEACHING OF ARITHMETIC

V. OBJECTS TO BE ACHIEVED IN ARITHMETIC 171

VI. NUMBERS AND NUMBER COMBINATIONS 191

VII. FUNDAMENTAL OPERATIONS ON WHOLE NUMBERS . . 221

VIII. COMMON FRACTIONS 249

IX. DECIMAL FRACTIONS 266

X. MEASUREMENT AND DENOMINATE NUMBERS 278

XI. PERCENTAGE 289

XII. THE SOLUTION OF PROBLEMS 294

XIII. DISCOUNT 329

XIV. PROFIT AND LOSS 336

XV. COMMISSION 341

XVI. INTEREST 344

CHAPTER PAGE
XVII. SAVINGS AND INVESTMENTS 357
XVIII. STOCKS AND BONDS 371
XIX. BANKS 384
XX. PAYING BILLS AT A DISTANCE 391
XXI. TAXES 395
XXII. INSURANCE 403
XXIII. GRAPHIC REPRESENTATION 412
XXIV. SQUARE AND CUBE ROOT 420
XXV. MENSURATION 426
XXVI. THE ARITHMETIC OF THE HOME 433
XXVII. ARITHMETIC FOR THE FARM 441
XXVIII. THE TEXTBOOK, ITS SELECTION AND USE 453
XXIX. MEASURING THE RESULTS OF TEACHING 468
INDEX 479

PART ONE

GENERAL PROBLEMS BEARING ON THE TEACHING OF ARITHMETIC

CHAPTER I

FORMAL DISCIPLINE

1. The purpose of this chapter. — Questions about the ends to be attained in education lead inevitably to a consideration of the doctrine of formal discipline.[1] It is the purpose of this chapter to present a statement of this doctrine, together with a brief résumé of the arguments that have been urged for and against it. This will serve to clear the ground and to lay a basis for the discussions which comprise the greater part of this book.

It is believed, moreover, that a review of this controversy, which has raged more or less acrimoniously for the last thirty years, may be of real service to the general reader. The advent into the world of an important new idea is greeted by many, especially by the youthful and enthusiastic, as a "revolutionary" event. The whole world is about to be upset; all that is old is "out of date," or at least insignificant when compared with the new. Later the old and the new are both seen in perspective and their relative significance more justly gauged. The debate on formal discipline, which has held a "close-up" on the educational stage for the last third of a century, now seems to be receding to a position in which its general bearing and significance are becoming more apparent.

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[1] "The problem of mental discipline, of determining under what conditions, by what methods, and to what extent training received in a given line of mental activity spreads to other lines of mental activity, is acknowledged to be the central problem of educational psychology." G. N. Whipple, in preface to *The Experimental Determination of Mental Discipline in School Studies;* by H. O. Rugg, Warwick and York, 1916.

In commenting upon this passage Moore says: "It is more than that, it is the central problem of educational philosophy as well, and the attitude which we who teach take upon this problem determines as nothing else does what we put into courses of study and how we teach that which we attempt to teach. Until we can get our bearings on this subject we simply can not get our educational bearings at all." E. C. Moore, *What the War Teaches about Education;* The Macmillan Company, 1919, p. 76.

"So many problems in the administration of schools, in the construction of courses of study, in the emphasis upon various aspects of school subjects, in fact, the ultimate values of education as a whole, depend primarily upon our attitude toward the problems of mental discipline and transference of training." Daniel Starch, *Educational Psychology;* The Macmillan Company, 1919, p. 191.

While the issue has by no means been settled, there now seems to be fairly general adherence to certain principles which are of far-reaching importance in dealing with many problems of teaching. For these reasons a chapter on formal discipline is placed at the beginning of this book.

2. The older theory in its simpler form. — Some decades ago there was much talk, as there still is in some quarters, of general intellectual ability or power. Analogy with physical strength or weakness was used freely. As some persons are of large and powerful physical build, while others are small and puny; so, it was said, there are some who have large and powerful intellects, while others have small and weak intellects. As the body may be strengthened by right living and exercise; so, it was assumed, the intellect may be built up and made more powerful by proper use.

The educational corollaries were obvious. Walking, running, or lifting heavy objects, though ever so useless in terms of the tasks performed, nevertheless serve to produce strong human forms — human forms with endurance not only in walking, running, or lifting heavy objects, but in performing hard bodily tasks of any kind. Similarly, it was said, any vigorous intellectual activity will develop strong intellects. It was believed that solving difficult problems in mathematics, translating intricate passages of Latin or Greek, and memorizing long pieces of literature would make the mind more effective — not only in solving mathematical problems, in translating ancient languages, and in committing poetry to memory, but also in performing difficult and arduous mental labor of any kind. Thus, education came to be regarded as a process for turning out intellectual athletes who could readily turn their prowess to meet any demands that life might make upon them.

3. The older theory in its analytical form. — It was but a step from this general point of view to a more analytical theory according to which the mind was believed to possess certain distinct faculties that might be developed separately, each by its appropriate training and exercise. The physical analogy still being adhered to, it was known that certain muscles might be developed by exercise while others were left flabby and weak. Only by well-balanced training can an all-round vigorous and powerful physique be attained. Similarly, it was thought, to secure an all-round strong and effective mind, all the "faculties" of the mind must be brought into use. Certain studies involving much memorizing were thought to strengthen the memory; other studies, such as mathematics, in which much reasoning or argumentation is involved, were thought to develop the reasoning faculties; and so on through the list. From this point of view, the requirements which should be met by a school curriculum were easily determined. Aside from developing the skill and furnishing the information needed in making a living and in performing the functions of a citizen, education should provide balanced training of the various faculties of the mind. The relative stress to be put upon the training of the faculties among themselves and upon information on the one hand and "mental training" on the other was, of course, a matter for debate, and the conclusion in any particular instance depended upon the temper and point of view of those making the decision. However, the elements supposed to be involved in each case were set forth clearly and definitely by the general theory, and it appeared that nothing more, either of psychology or of educational doctrine, could be required in this particular field.

4. Modern criticism of the faculty theory. — This camp of smug and satisfied psychologists and educational doctrinaires was thrown into utter confusion by the modern critic and experimenter. Upon closer analysis it appeared that it would be necessary to revise, if not entirely to abandon, the doctrine of the "faculties of the mind." Indeed, it requires no profound investigation to show, for example, that in carrying through a difficult piece of work in reasoning, the memory is involved equally with the so-called reasoning faculties. To reason either inductively or deductively, the material with which the mind works must be seen floating visibly in the memory; that is, a large number of facts — in some cases a very large number — must be memorized so as to present themselves without difficulty. Imagination, too, plays its part. To carry out the figure: As the various elements to be considered (in inductive reasoning for instance) are floating visibly in the memory, they present a variety of combinations from time to time, and it is necessary to seize these combinations and retain them in memory for further and more leisurely contemplation.

Again, it is easily shown that in practical life ordinary acts of the memory are equally complex. Why is it that an actor can memorize a whole play containing some thirty thousand words with no more effort than would be required for him to memorize a few hundred nonsense syllables? Compare the difficulties in memorizing a ten-place decimal number, a sentence consisting of ten words, and the spelling of a word with ten letters. But you say the reasons are obvious. Of course they are obvious. But obvious as they are, they serve to point out what is not always sufficiently recognized even nowadays — namely, that normal acts of remembering, such as are required of us in our daily lives, are so complex, are so different from mere non-associative retention, like that needed in recalling sequences of numerals or nonsense syllables, that even great ability in the latter may be of small significance in measuring our really effective memory power. So it appears that the so-called faculties of the mind are inextricably intermixed in every mental act except possibly the very simplest. Considerations such as these have rendered untenable the old doctrine of the "faculties" and the educational theories based upon it, in so far as these can not be shown to have other foundations.

5. Methods of the experimental psychologist. — The experimental psychologist has approached the problem from a different angle. The question which he seeks to answer by actual tests is whether training in one mental ability develops other and different abilities.[1] It is far beyond the scope of this chapter to attempt an adequate account of the large number of experiments that have been performed or to give any but a very general evaluation of their results. Rugg[2] gives a description of thirty such experiments. Somewhat shorter lists are described by Starch[3] and by Coover.[4]

The method used in one of these experiments indicates the general method used in all of them.

In 1916, Coover[5] reported the following experiment, which was performed in 1905:

Two reagents were trained for eleven days in marking out words containing *e* and *s* in selected columns of *The Outlook*. Each reagent looked over 12,000 words in each day's practice.

Tests were taken before and after training in marking out: (1) Words in *Outlook* columns containing *e* and *s*, *i* and *t*, *s* and *p*, *c* and *a*, *e* and *r*. (2) Words on manuscript pages, containing

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[1] The meaning of "ability" and "different abilities" is considered in sections 9, 10.

[2] Rugg, *op. cit.* (p. 3), pp. 7–24.

[3] Starch, *op. cit.* (p. 3), pp. 191–213.

[4] J. E. Coover, "Formal Discipline from the Standpoint of Experimental Psychology"; *Psychological Monographs*, 1916, Vol. XX, No. 3.

[5] *Ibid.*, pp. 34–42.

a and *n*, *l* and *o*, *e* and *r*. (3) Common nouns in *Outlook* columns. (4) Words in *Outlook* columns containing *e* and *s*.

Training was given in crossing out words containing *e* and *s*. After the training, the speeds of crossing out the words containing *e* and *s* were increased by 133 per cent and 45 per cent respectively. That is, the speeds of the final test for the two reagents were 233 per cent and 145 per cent of the speeds of the first test. The speeds in crossing out words containing *i* and *t*, *s* and *p*, *c* and *a*, *e* and *r* were increased by 100 per cent and 43 per cent; the speeds in crossing out words in manuscript pages containing *a* and *n*, *l* and *o*, *e* and *r* were increased by 42 per cent and 25 per cent; the speeds in crossing out nouns were increased by 16 per cent and 7 per cent; and the speeds in crossing out words containing *e* and *s* (different material being used from that used in the original test) were increased by 76 per cent and 48 per cent.

This experiment shows large increases in speed in the operations on which no training was had. It is not safe, however, to conclude that such increase was due entirely to training in crossing out words containing *e* and *s*. If this experiment had been performed later, after the technique became more fully developed, there would have been a "control experiment" in which reagents would be given the first and the last tests but no training in crossing out words containing *e* and *s*. Experience has shown that the reagents of the "control experiments" would gain considerably though not nearly so much as those who are trained.

6. Spread of ability developed by laboratory training. — Two distinct types of attempts have been made to measure the effect of the training of one ability upon supposedly different abilities. One of these types is represented by the experiment described in the preceding section. An example of the second type is given in section 14. The experiments of the first type consist essentially in giving training, specially arranged for the purpose of the experiment, in a definite ability and then testing the effect of this training upon different abilities.

The following list of experiments, taken largely from Coover,[1] shows fairly well the field which the experimental psychologist has thus far attempted to cover in the laboratory and affords a basis for intelligent consideration of the significance of his results. The experiments are given in the chronological order in which they were performed.

1. James (1890): Practice in learning one poem produced little or no gain in ability to learn another poem.
2. Jastrow and Cairnes (1891–2): Two mental processes carried on at the same time, such as finger movement involving rhythm and reading or adding, interfered with one another except that maximum rate of beating hurried the mental process.
3. Bergstrom (1892–3): In sorting a pack of cards into compartments and then sorting them into the same compartments differently arranged, it was found that the first process interfered with the second.
4. Angell and Moore (1896): Practice in reacting to sound stimuli lowered the time required for reaction to visual stimuli.
5. Epstein (1896): Sensitivity of one sense was increased when another sense was simultaneously stimulated. Thus vision was keener when the sense of hearing was stimulated. The same result had been obtained earlier by Urbantschitsch.
6. Gilbert and Fraker (1897): Training in simple reaction, or in reaction with discrimination, to sound stimuli caused a decrease in the time required for simple reaction, or for reaction with discrimination, to light, electrical, and tactual stimuli.
7. Vogt (1899–1900): Training in disregarding one kind of distraction in doing one kind of work enabled one better to disregard another kind of distraction while doing the same or another kind of work.
8. Thorndike and Woodworth (1901): Training in estimating areas and weights that were within certain limits improved the ability to estimate areas and weights not within these limits.
9. Bair (1902): Practice in repeating the letters of the alphabet either

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[1] *Op. cit.* (p. 7).

forward or backward with interpolation of one letter between each pair of successive letters increased the ability to repeat the same series with the interpolation of a different letter. Similar results were obtained from practice on the typewriter.

10. Ebert and Meumann (1904): Memorizing nonsense syllables improved the memory for letters, numbers, words, nonsense syllables, Italian words, verses of poetry, lines of philosophic prose, and optical symbols.

11. Liddle (1904) (Reported by Coover): Practice in sorting cards of one set of colors or of one set of geometric forms increased the ability to sort cards of another set of colors or of another set of geometric forms.

12. Bennett (1907): Practice in discriminating between shades of blue improved the ability to discriminate between shades of other colors and also to discriminate pitch in music.

13. Coover and Angell (1907): Practice in discriminating intensities of sounds increased the ability to discriminate intensities of light.

14. Judd and Cowling (1907): Drawing figures with eyes open improved the ability to draw the same figures with the eyes closed.

15. Book (1908): Practice in sight typewriting increased the ability to learn touch typewriting.

16. Fraker (1908): Memorizing series of sounds improved memory of series of grays, tones, pitches, and verses of poetry.

17. Schalckow and Judd (19c8): Described in section 13.

18. Winch (1908): Practice in memorizing poetry improved the ability to memorize historical prose and geographical passages.

19. Dearborn (1909): Repeated the experiment of Ebert and Meumann (No. 10 above) using a control experiment and found a smaller gain due to training only.

20. Kline (1909): Practice in cancelling *e*'s and *t*'s on pages of prose decreased the ability to cancel nouns, verbs, prepositions, pronouns, and adverbs.

21. Gamble (1909): Practice in memorizing colors improved the ability to memorize odors and nonsense syllables.

22. Dunlap and Wells (1910): Reaction to one of two simultaneous stimuli was different from the time of either alone, being shorter to a visual stimulus when a sound stimulus was also given than to a visual stimulus alone and longer to a sound stimulus when both were given than to a sound stimulus alone.

23. Ruger (1910): Practice in solving puzzles improved the ability to solve other puzzles.

24. Wallin (1910): Practice in using one eye in unusual ways improved the ability to use the other eye in the same ways.

25. Sleight (1911): Training of memory upon matters similar to school work resulted in a few significant gains in memory for material similar to and different from that used in training and deteriorated memory for some such material.

26. Rall (1912): Memorizing poetry or prose in English or in a foreign language or memorizing irregular verbs or vocabularies improved memory for poetry and for nonsense syllables.

27. Coover (1916): Experiments designed to measure attention and the transfer of attention seemed to show that the power of sustained attention is developed by practice.

28. Webb (1917): Practice in learning mazes greatly improved the ability to learn other mazes. This was true both of human beings and of rats. Other experiments are described more fully in section 13.

7. General characteristics of laboratory training experiments. — Two essential characteristics stand out prominently in the experiments described in the preceding section. The first of these is the simplicity or narrowness of the training whose transfer or spread has been studied. Committing prose or poetry to memory; memorizing sequences of sounds, colors, or nonsense syllables; sorting cards; crossing out certain parts of speech, or words containing certain letters — such are the abilities in which training has been given with a view to determine its effect on other abilities.

The second characteristic is the similarity of the abilities in which training was given and the abilities to which transfer was studied. The transfer of the results of training in one kind of memorizing to another and similar kind of memorizing; in the discrimination of shades of one color to the discrimination of shades of another color, or to pitch in music; in sorting cards of one kind to the sorting of cards of a different kind or to reactions on a typewriter — these are fair representations of the kinds of transfer that have been studied. (Cf. Rugg cited on pages 32, 33.)

These characteristics which greatly limit the significance of these investigations are no doubt due to the fact that the tracing of the effects of complex training into remoter fields would present a hopelessly complicated problem. A problem, such as finding the effect of producing a superior doctor's dissertation on some topic, say in education, upon ability to organize the selling force of an automobile company — to take an instance at the opposite extreme from those studied in the laboratory — would be vastly more interesting and more significant than those enumerated above, but it would be practically impossible to carry out the necessary tests.[1]

8. Amount and character of the spread or transfer. — Nearly all the experiments show a spread or transfer of the effects of training, either positive or negative. In some cases, training in one ability interferes with another ability (see Nos. 3, 20, of section 6), but in by far the greater number of cases, training in one ability increases other related abilities. The amount of transfer is important and ranges all the way from a small negative quantity to nearly complete positive transfer. In the majority of cases, the increase in ability in the nonpracticed operation ranges, according to Coover,[2] from 25 per cent to 75 per cent of the increase of ability in the practiced operation. The transfer is greatest when the similarity between the practiced and the nonpracticed operations is great and decreases rapidly as this similarity decreases. The amount of transfer also depends upon the manner in which the practiced operation is learned (see section 13).

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[1] "Picture for a moment the difficult situation before the investigator who attempts, *e.g.* to determine what training effect a year's study of high-school algebra has upon certain mental abilities. His subjects are statistically a heterogeneous class, of various ages, of various previous trainings in mathematics, registered in diverse courses or curriculums, having a variety of daily interests and taking part in a variety of daily activities both in school and out. If he has many subjects, they are necessarily distributed in several sections under different instructors, and thus subjected in their training to different organizations of the material in the training course itself and to diverse methods of teaching. He further faces the difficulty of selecting a mental ability trained by his training course, but which other concurrently taken courses do not also train to an appreciable extent. Again, granted that the desired mental ability has been isolated for work, our investigator must next devise a series of mental tests which will be a real measure of efficiency in the selected ability. This, in itself, is no small task, and may well be the first point of attack by critics of such investigations." Rugg, *op. cit.* (p. 3), pp. 25–26.

[2] *Op. cit.* (p. 7).

9. Ability, industry, enterprise, success. — There is no source of confusion more prolific than ambiguity of words. What, for instance, is meant by *ability*, and by *different abilities?* We are, on the one hand, bound to use words as nearly as possible with their dictionary meanings (the meanings given them by current usage); and on the other hand, we are obliged so to delimit their meanings that they may fairly represent the distinctions which it is desired to make.

Illustrations will help to show the essential meanings which it is proposed to attach to the words which head this section: A tramp may be induced to saw wood if to-day's dinner can be procured in no other way. The provident farmer will saw wood in order that it may have time to dry and his meals may be cooked a year hence. No such distant prospect would stir the tramp to action. In a certain sense, the tramp and the farmer are equally capable of sawing wood while the same stimulus would produce entirely different responses in the two men.

The tramp and the farmer both have the *ability* to saw wood, but the farmer shows the additional element of *industry* by sawing wood that is not needed for a long time hence.

Two farmers living side by side need wood for the winter. One of them cuts it in the good old-fashioned way, while the other rigs up a power plant, using a stream that flows over his land as the source of power. Incidentally he uses the power for many other purposes besides sawing wood. The second farmer shows by this act that he possesses something which the other does not have (we assume that they have equal opportunities to rig up the power plant). This something we shall call *enterprise* (in rigging up the power plant).

We shall now hazard formal definitions of the words *ability, industry, effectiveness*, and *success*.[1]

A person is *able to do a certain act* if there exist stimuli which, together with the removal of possible volitional inhibitions, will make him do it (not merely attempt it).[2]

A person is possessed of *industriousness in doing a certain act* in proportion as stimuli which cause normal persons to perform this act cause him to apply himself in doing it more or less nearly to the limit of his ability (we assume again that possible volitional inhibitions are removed).

A person having the ability to do a certain act is said to be *effective in doing it* in proportion to his industriousness in doing it.

A person is said to be *enterprising in doing a certain act* in proportion as he seeks out and uses the best ways of doing it.

A person is *successful in doing a certain act* in proportion as he is effective and enterprising in doing it.

It is seen that *ability to do a certain act*, as here defined, is an absolute term while *industriousness*, *effectiveness*, *enterprise*, and *success* are relative terms. Where ability exists, the degree of effectiveness varies directly with industriousness, while success varies jointly with effectiveness and enterprise.

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[1] Definitions of words, other than purely technical mathematical terms, are unsatisfactory, especially to a mathematician. They fail in definiteness, that is, in clearly delimiting the class of things (entities, qualities, etc.) to be represented by each word. The reason for this is that the world, other than its mathematical abstractions, is so complex that we are not able fully to unravel it and separate its infinities of objects of thought (things, entities, qualities, etc.) into well-defined classes. Words, as they are commonly used, often overlap even when they are supposed to represent mutually exclusive classes. The reader should bear this general fact in mind in considering the definitions given here.

[2] This definition includes the elements of time, quantity, and quality. It does not specify *ability to run* but *ability to run a certain distance in a given time and under given conditions*. It does not specify *ability to dance* but *ability to dance in a certain way*.

Clearly aware of the objections that may be urged, the writer ventures the following equations:

Ability plus industriousness equals effectiveness.
Effectiveness plus enterprise equals success.

If interpreted sympathetically, it is believed that this analysis will be useful for the purpose of the discussion which follows.[1]

10. Analysis of ability. — It is probably true that every specific ability may be regarded as composite.[2] An instance of a highly composite ability will illustrate: Not long since, in company with a man interested in the subject matter of this chapter, the writer drove an automobile over one of our great National highways. The road was strange to the driver and had to be picked out at each doubtful turn or corner by the aid of a map placed before him and by the markings of the road on posts and buildings. Besides driving the car and picking out the road, the writer was engaged in a lively discussion of formal discipline. The highly composite character of this act is best appreciated by recalling the process of learning some of the many separate acts (each in itself highly composite, as was clearly enough realized at the time of learning it) of which it was composed. Years ago as an absolute novice at driving he was occupied to his very limit in steering a car so as to keep it from going off a flat, wide road. Both hands gripped the wheel nervously and his eyes were kept constantly on the road, while the speed of the car was regulated by an experienced driver seated at his side. A curve in the road was a serious matter, though the speed of the car never exceeded fifteen miles an hour. As separate and not very simple acts came the shifting of gears, stopping and starting, and going forward and backing without killing the engine. The proper and timely application of brakes was a matter of no small moment. Learning to take curves in the best way required thought and many separate trials.

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[1] The final answer to criticisms of these definitions must be that one has the right (logically) to define words any way one pleases, the only obligation being that he must adhere to the definitions adopted throughout the discussion. The effort has been made, however, to make them conform as nearly as possible to current usage.

[2] It is not pertinent here to go into a detailed discussion of the question whether an act, such as the involuntary shutting of the eyes under the stimulus of a sudden flash of light, is a simple or a composite act. Obviously the answer would turn upon the exact use of the words "simple" and "composite" and for the purpose of the present discussion would be of no importance.

The customs of the road had to be brought to mind and partly learned for the first time, the markings of the particular highway had to be learned and skill acquired in noting them quickly and in time to take the proper turns. Finally, all these abilities had to be synthesized into one ability — the effective driving of the car over this particular highway. This constituted one part of the composite act now being described. The other part — the discussion of the doctrine of formal discipline — had its foundation on the driver's part in at least two decades of attention to the subject, in the reading of innumerable books, monographs, and notes (not to consider at all learning to read, the partial mastery of several languages, and what not), the study of the results of numerous experiments, listening to many lectures and discussions, informal observation of facts with which one comes in contact in a long period of years of fairly checkered experience, reflection and analysis such as one is likely to make when one has an abiding interest in a subject. All of these were brought into play and formed part of the *ability* to do this composite act of driving and thinking and talking. The refinement of the analysis could proceed *ad infinitum*. This illustrates what may be called a hierarchy of abilities or of techniques.

On this subject, Angell[1] remarks:

> One kind of process, which certainly goes on all the time and which may have a remote bearing on the general point at issue, is the incorporation of

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[1] James Rowland Angell, in a symposium on "The Doctrine of Formal Discipline in the Light of the Principles of Psychology," *Educational Review*, Vol. 36, 1908, pp. 5, 6.

smaller habits in larger habit groups. The child in learning to write has at first to give all his energy to the mere grasping and guiding of the pen. As dexterity is gained, the movement gradually comes to take care of itself and gets incorporated in another and larger coördination, *i.e.* the spelling-and-writing coördination. This in turn gets taken up into a "paragraph-construction habit," which in its own turn may be swallowed up by the chapter, article, or section habit. Not that this account necessarily follows any unchanging chronological sequence or is true of all persons, but that it illustrates what is generally true all along the line, *i.e.* that specific habits are constantly merged with other specific habits to furnish forth larger and more complex coördinations. . . . Evidently it might well often occur that a habit acquired in some special study should find a place in a larger group of habits apparently quite disconnected from the study. This is peculiarly true, it may be added, of all the common studies of the elementary school.

It seems clear, too, that habits closely akin to one another may readily reinforce each other in a practical way, even though the literal fact should prove to be that one or other is slightly modified in this case, rather than merely reinforced. For instance, a boy who has learned to play baseball and to judge accurately the position of a ball in the air has a large part of the difficulty of certain strokes in tennis already conquered, despite the fact that the position which he must assume to meet the ball in the two cases is somewhat different.

The analysis of an ability suggested by the above should be of assistance in making clear what is meant by "common elements" or "common factor" in different abilities. A discussion of this "common factor" is given in the next section.

11. Distinction between transfer and spread of the effects of training. — Fraker, Coover, and others have developed analyses of the "common factor," which seem useful and significant. To quote Coover:[1]

It is generally agreed that the "common factor" is responsible for the general effect of special practice; but as to what it is or may be, there is difference of opinion. It is interesting to note that its nature is growing more complex as experiment and discussion advance. In the early days it was regarded as largely physiological and related simply and immediately to the data of presentation, or, as mental habit, related definitely to the method of the experiment. . . . And "general effect" was said to decrease directly with decrease in similarity of matter and method between the two tests. . . .

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[1] *Op. cit.* (p. 7), pp. 224 ff.

More recently the "common factor" has been found to be made up of subjective factors; not altogether, but largely, independent of the matter and method of the tests (admirably shown by individual variation) and to be anything but simple. . . .

Ibid., p. 230: When the more orderly changes in processes effected by practice in one task are carried over to another, we have what the writer [Coover] thinks is best described as a "spread of training." . . . The common factor" appears to be *formal* and to be relatitvely detached from the data of presentation and [in some experiments] from method. No systems of imagery were carried over. The practice-effect consisted in stripping the essential process of unessential factors, in *facility* [the italics are mine] in developing automatic coördinations, in establishing habits of higher order, in dealing more effectively with vague imagery, etc. This kind of general effect seems to rest more directly upon *modes of mental processes* than upon the material of experience and seems capable of description largely in terms of emotional and volitional attitudes and of the control of attention.

The term "transference" seems particularly applicable to the carrying over, from one test to another, of the *material of experience*. It takes place typically in the "adaptation" to a novel exercise and in the application of physiological processes and systems of imagery in an exercise similar to the exercise in which they were developed.

These two kinds of general effects of special practice bear to each other a relation analogous to form and matter in logic; and although inseparable, should be discriminated for the purpose of prohibiting [preventing] the limit of search for the "common factor" to the case of "transference" alone.

Under "form of experience" whose carrying over is termed "spread," Coover enumerates:[1]

1. Problem (interpretation of instructions). [It is a common experience that untutored persons try to answer questions without apprehending definitely what the questions are. What teacher of mathematics has not witnessed attempts to solve problems without knowing what they were! The writer once saw a case of tears over the equation, $\sqrt{x^2} = x$.]

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[1] *Op. cit.* (p. 7), p. 233.

2. Attitude (emotional, volitional). [For remarks bearing on this point, see section 27.]
3. Ideals (controlling concepts). [See section 32.]
4. Purpose (definition of volitional attitude).
5. Intellectual processes: *a.* elaboration of method, *b.* short-circuiting of processes, *c.* higher order of control, *d.* elimination of accompaniments of discrimination, cognition, movement, reproduction, *e.* growth in simplicity or complexity of imagery, *f.* coördination of part processes.
6. Control of attention: *a.* in degree, *b.* in direction, *c.* in distribution, *d.* in quickness of adaptation, *e.* in duration of concentration.

[These are the elements which have received the least attention at the hands of the experimenters while, at the same time, they are no doubt of the greatest significance for the solution of the problem of formal discipline.]

Under "material of experience" whose carrying over is termed "transfer," Coover enumerates:

1. Simple imagery (visual, auditory, kinæsthetic, tactual, etc.): *a.* direct, *b.* associative [(1) in the same sense-mode, (2) in a different sense-mode].
2. Compound imagery (of the above): *a.* direct, *b.* associative.
3. Complex imagery (spatial, temporal, causal, histrionic): *a.* direct, *b.* associative.
4. Systems of imagery (representative schemes, mnemonic devices).

Many of the elements enumerated are obvious to the general reader; others appear only upon insistent analysis, and especially upon a detailed study of experiments on the carrying over of the effects of training. To attempt to put the matter briefly in our own words:

Training develops a certain ability of the mind so to *orient itself* with respect to a problem as to make success more likely. Learning not to react in the usual way to stimuli inciting diversions is a good instance. This effect of special training on more general abilities is called "spread" of the effects of training.

Special training develops certain habits of automatic reactions and furnishes the mind with a certain set of imagery. The carrying over of such habits and imagery to other acts is called "transfer" of the effects of training.

12. What elements are carried over? — There is a wide range of opinion as to just what elements in learning are carried over. It is agreed that in case two processes of learning or two abilities have "elements in common" then such elements are carried over, but there is no complete agreement as to what constitutes "common elements."[1]

In general, the experimenter in the psychological laboratory is inclined to a view limiting the "common elements" rather closely, while the general psychologist is inclined to the opposite view, and that in spite of the fact that the experimental results are equally known to both. This may be due to the fact that those who approach the problem primarily, if not exclusively, from the laboratory point of view are more inclined to regard the mind as amenable to laboratory investigation, and as (for the sake of the argument at least) simpler than it really is. They are constantly endeavoring to explain the workings of the mind in terms of fairly simple stimulus-reaction relations. The real explanation perhaps lies in this, that the experimental psychologist is by force of circumstances occupied with the simple peripheral activities of the nervous system rather than with the elements of reflection and judgment which have their seat mainly in the central cortex, and there are good grounds for believing that the latter are the elements which carry over most largely.

A molecule consisting of several hundred atoms is bad enough for the chemist to handle. Experimental psychology is yet in its infancy; and when we consider the countless millions of atoms and molecules at work in the nervous system and the corresponding complexity of the workings of the mind, the layman, and even the general psychologist, may perhaps be pardoned if he does not base his conclusions too exclusively on the results of the technical laboratory as over against the results of that great world laboratory, in which the workings of the mind as a whole have been observed for ages and in which the qualities of the mind are inferred from its more general manifestations. This view does not disparage detailed experimental work or express pessimism as to its value, but it does counsel modesty in interpreting results. Doubtless there is a mathematical formula which represents the totality of mental phenomena, but its mode of expression may yet be a long, long time in formulation.[1]

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[1] See passage from Coover just cited.

In a symposium on "The Doctrine of Formal Discipline in the Light of the Principles of General Psychology,"[2] Angell says:

> It should not be assumed that because the faculty psychology is exploded, therefore the inferences based upon it are all essentially erroneous. They may have other foundations than those on which they are supposed to rest. . . .
>
> The persistent and voluntarily directed use of attention, especially when the subject attended to is lacking in interest, speedily becomes acutely distasteful. Voluntary attention involves some strain, and this strain, if long continued, is certain to become unpleasant. . . . Now no small part of the discipline which comes from the effortful use of attention in any direction and on any topic is to be found in the habituation which is afforded in neglecting or otherwise suppressing unpleasant or distracting sensations. We learn to "stand it," in short. . . .
>
> Any one can attend to things which interest or please him as long as his physical strength holds out. But to attend in the face of difficulties which are not entertaining is distinctly an acquired taste, one to which children and primitive peoples always strenuously object. From this point of view it may be well that such studies as the classics and certain forms of mathematics have a peculiar value in affording the maximum of unpleasantness diluted with a minimum of native interest, so that a student who learns to tolerate prolonged attending to their intricacies may find almost any other

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[1] "We have not, however, the knowledge necessary to state even a single one of the simplest mental functions in terms of the physiological conditions whose effects on behavior it identifies. We know that ability to spell *cat* means in the last analysis some actual bonds between neurones, but we do not know what bonds or between what neurones. Our analyses have progressed toward, but are nowhere near, that limit." Edward L. Thorndike, *Educational Psychology;* Teachers College, Vol. III, p. 66.

[2] *Op. cit.* (p. 16), pp. 3 ff.

undertaking by contrast easy and grateful. . . . Certain it is that something of the sort occurs and that it is an acquirement which may presumably be carried over from one type of occupation to another. . . .

Again it is held by certain psychologists that, although each form of sensory and ideational attention involves a special and peculiar motor attitude not found in any other form in which attention may be exercised, it is nevertheless true that there is a general attitude on which each of these special forms is grafted which remains as a constant background for all. Of course if this contention be true, and I am disposed both on theoretical and on experimental grounds to think that it is, there would be some matrix common to all acts of attention, and any development whatever would affect this central core in some degree. . . .

Ibid., p. 10: We may notice, however, that memory is a function of concentrated attention to the extent at least that, other things equal, the person who possesses the most concentration of attention will be found most tenacious of material learned and most prompt and effective in commanding this material when wanted. This consideration would lead us to expect that almost any training of memory would show some detectable effect in any subsequent use of the memory processes. Recent experiments indicate the correctness of this anticipation.

It should not be forgotten that a very real intellectual advantage is gained from any well-organized study, in that one is given a vivid illustration, which may prove contagious if the teaching be well done, of the possibilities of method and technique in thinking. The leverage given by system and organization is thus made clear. The precise system appropriate to a given problem may, however, be quite inadequate to some other problem, so that the profit on this score is not without its limitation.

It has been maintained that, after all, the great advantage in any serious study — the formal disciplines as well as others — is in the creation therefrom of certain ideals which are as such applicable to almost any situation. Such ideals are thoroughness, accuracy, system, and the like. I believe this contention may be granted without argument.

But in all these cases, the instances of inhibition as well as those of reinforcement and incorporation, it seems probable that a certain gain in the power to use and sustain attention will accrue from any purposeful and persistent intellectual application. This result may be expected to come in part from the suppressing or disregarding of disagreeable and distracting sensations, and in part from the discipline afforded to the common element in all acts of attention. . . . This principle probably holds true in memory, in reasoning, in observation, and in all the forms of mental activity which common thought and language distinguish.

In the same symposium, Pillsbury says:

> *Ibid.*, p. 26: Habits of attention in general and to one kind of material not to another are undoubtedly acquired through study of any kind. Even the habit of using books intelligently needs to be acquired in the early stages; and once acquired, can be transferred to other fields. Even more important is the capacity for selecting the important points and in properly knitting them to the relative facts, to the facts and occasions that render their recall desirable. For most adult learning it is essential to remember the fact apart from the language in which it is expressed and apart from the particular connections in which it is first learned. All these habits of easy and effective learning can be acquired by learning any sort of material that it is important to remember; and once acquired, may be transferred to almost any other field.
>
> In both rote and logical learning there are definite habits and capacities of attending to be acquired, and these may apparently be acquired in one field and used in another. We have to do in memory, then, with a large number of fairly distinct physiological capacities, but their use has become so dependent upon habits common to the different capacities that they are functionally parts of a common whole.

And finally, in the same symposium, Judd says:

> *Ibid.*, p. 33: All the facts thus argue, I firmly believe, not for a discreteness of mental functions, but rather for a unity and compactness of mental life such that, if you influence one phase of a man's conscious being, you contribute, sometimes negatively to be sure, but none the less surely, to all the different elements of his nature. . . .
>
> A second type of illustration is to be found in the fact that we all know what is meant by the phrase, the scientist's attitude. The man who, through long training in the analysis of situations, has acquired certain general modes of intellectual procedure will show himself a scientist in the presence of any emergency, however novel. Every new situation is attacked in the fashion for which his training has prepared him.

This same general topic is considered further in the next section.

13. Manner of learning related to transfer. — It is certain, both from experimental evidence and from facts within the reach of all, that ideas carry over more largely than mere skill not connected with ideas. When that which is learned is generalized into a concept, it acquires a wider bearing and therefore spreads more readily to other activities some of which may be only remotely similar to those in connection with which the learning took place.[1]

The following experiment illustrates this important point:

Carrie R. Squire made an experiment regarding the transfer of neatness: "At the Montana State Normal College careful experiments were undertaken to determine whether the habit of producing neat papers in arithmetic will function with reference to neat written work in other studies; the tests were confined to the intermediate grades. The results are almost startling in their failure to show the slightest improvement in language and spelling papers, although the improvement in the arithmetic papers was noticeable from the very first." Bagley, *The Educative Process*, p. 208.

This experiment was repeated under the direction of Ruediger with the difference that along with the specific training in neatness in one particular study, a general practice of neatness in daily life was held up before the class as an ideal to strive for. Care was taken not to discuss neatness in the other classes. Two schools which showed an appreciable improvement in neatness in the study where special training was given also showed considerable, though less, improvement in the other studies.[1]

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[1] Nellie P. Hewins states: "Ruger in testing the transfer of specific motor habits, in his study of the Psychology of Efficiency, had his subject from the psychological laboratory use 'animal' and 'human' methods in taking apart or putting together various puzzles. The preliminary and final tests consisted in testing a subject with a puzzle thrown in chance positions. The training consisted in handling four special but important positions. The training of another subject consisted entirely of chance positions in a series about half the length of the first subject's series. The second tests of the first subject showed no improvement over the initial results and were inferior to those of the second subject. 'This failure to profit by the highly specialized training seems to have been due to lack of a generalized rule of procedure.' Ruger therefore concluded that 'in general the value of specific habits under a change of conditions depended directly on the presence of a general idea which would serve for their control.' In regard to concrete imagery he found that 'the value of the image as well as of the motor habit depended on the precision of the analysis'." *The Doctrine of Formal Discipline in the Light of Experimental Investigation;* Warwick and York, 1916, p. 23.

For Ruger's memoir, see "The Psychology of Efficiency," *Archives of Psychology*, No. 15, June, 1910, Columbia Contributions to Philosophy and Psychology, XIX, 2.

"The first and most striking fact which is to be drawn from school experience is that one and the same subject matter may be employed with one and the same student with wholly different effects, according to the mode of presentation. If the lesson is presented in one fashion, it will produce a very large transfer; whereas if it is presented in an entirely different fashion, it will be utterly barren of results for other phases of mental life." Charles Hubbard Judd, *Psychology of High School Studies;* Ginn and Company, 1915, p. 412.

Judd[2] reports the following (No. 17 of section 6):

Two groups of pupils in the fifth and sixth grades were required to hit with a small dart a target which was placed under water. The difficulty of hitting the target arises, of course, from the deflection which the light suffers through refraction. The target is not where it seems to be, and the boy must fit his aim with the dart to conditions which differ from those which he knows in ordinary life. The amount of refraction and the consequent displacement of the target are capable of definite theoretical explanation before one throws the dart. In this experiment one group of boys was given a full theoretical explanation of refraction. The other group of boys was left to work out experience without theoretical training. Those two groups began practice with the target under twelve inches of water. It is a very striking fact that in the first series of trials the boys who knew the theory of refraction and those who did not gave about the same results. That is, theory seemed to be of no value in the first tests. All the boys had to learn how to use the dart, and theory proved to be no substitute for practice. At this point the conditions were changed. The twelve inches of water were reduced to four. The difference between the two groups of boys now came out very strikingly. The boys without theory were very much confused. The practice gained with twelve inches of water did not help them with four inches. Their errors were large and persistent. On the other hand, the boys who had the theory fitted themselves to four inches very rapidly. Their theory evidently helped them to see the reason why they must not apply the twelve-inch habit to four inches of water.

Judd also reports the following:

Ibid., p. 30: Two observers were given a series of tests in the comparison of two geometrical figures. The figures compared were complex and were incorrectly perceived because of their complexity, giving rise to what is known as geometrical illusion. One of the two figures was overestimated, the other was underestimated. As a result of a long series of comparisons, the two observers ultimately overcame the tendencies toward overestimation and underestimation, that is, they learned to apprehend the lines correctly. They both learned this lesson in about the same number of comparisons, showing that they were both at the outset equally capable of taking on the effects of practice. During the course of the experiment one observer was kept in total ignorance of the results of practice, while the other was fully informed. Thus when they entered upon the second stage of the experiment, one had practice but did not know its effects upon him. The other had practice and did know its effects. The figures which they were using for comparison were reversed, and a second series of tests began. When they began working with the reversed figures, both observers showed confusion under the new conditions. Very soon, however, the observer who knew about the effects of practice adjusted himself to the new demands and rapidly overcame the illusion. There was in his case a speedy and advantageous transfer of practice. The other observer who did not know the effects of his earlier experience showed a greater error than at any time in the first series; and what is still more important, he showed no disposition to improve.

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[1] W. C. Ruediger, "The Indirect Improvement of Mental Functions through Ideals," *Educational Review;* 1908, pp. 364–371.

[2] *Op. cit.* (p. 16), pp. 36, 37.

Ruger[1] remarks in reporting his experiments on learning to solve puzzles:

> In some cases a generalized formula developed in connection with the first case was essential to effective transfer of motor habits to later modifications of the first case.
>
> Transfer was more effective in these cases where the formula or general rule was developed in the first few trials and where the formation of perceptual motor habits had been controlled and interpenetrated by it from the start, than when the generalization had been arrived at after those habits had been set up.

14. Transfer and spread of effects of school courses. — Efforts have been made to study the effects of courses of instruction given in the schools both upon success in other courses of study and upon special abilities connected more or less closely with the subject matter of those courses. Studies have been made of the correlation of successes in various studies, such as English and arithmetic in the grades or Latin and English in the high schools. The results are complicated and not of particular significance in the study of transfer or spread in the sense these are considered here. Exceptions may be noted in such cases as in the effect of the study of Latin upon the understanding of certain English words, etc. In most cases, transfer or spread being admitted, it is difficult, if not impossible, to determine in what direction the carrying over has taken place, even when the correlation is high. Have the studies *A* and *B* been mutually beneficial? Has *A* been beneficial to *B* and not *B* to *A*, or *B* to *A* and not *A* to *B*, or have they been mutually antagonistic? To these questions the most elaborate tabulations of comparative grades fail to give conclusive answers.

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[1] *Op. cit.* (p. 24), p. 88.

Below is described in some detail an investigation by H. O. Rugg, on the carrying over into simple tests of the results of training in a school course.[1] The problem is stated by Rugg as follows: "The problem is to investigate the effect of a semester's training in descriptive geometry upon specific abilities in the mental manipulation of spatial elements, (*a*) of a strictly geometrical type; (*b*) of the quasi-geometrical type; (*c*) of a nongeometrical type." The data were secured from 413 students in the University of Illinois during the years 1913 and 1914. Of these, 326 took a one-semester course in descriptive geometry. The "control group" consisting of 15 students in the College of Engineering and 72 in the School of Education did not take the course. Before beginning the course in descriptive geometry and also after its close, all subjects took six tests designed to measure their ability in mental manipulation of spatial elements both geometrical and nongeometrical. These tests were as follows:

Test 1. Ten examples like "Divide eighty-one by seven." (Time: 60 sec.)

Test 2. Ten examples like "Divide eight sixty-two by three." (Time: 90 sec.)

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[1] *Op. cit.* (p. 3), pp. 75 ff.

Test 3. Straight-line Alphabet Test. (Time: 75 sec.) These words were given:
cat | pen | jury | does | rank | this | why | girl | about | lapse | him | stead |
Picture each word on this sheet in the straight-line alphabet and write below each word the total number of continuous straight lines (strokes) which would be required to print the word, using this style of letter.

[The straight-line alphabet was shown on a blackboard.]

Test 4. The Painted Cube Test. (Time: 80 sec.)

A three-inch cube, painted on all sides, is cut into one-inch cubes.

1. How many one-inch cubes have paint on three sides?
2. How many one-inch cubes have paint on two sides?
3. How many one-inch cubes have paint on one side?
4. How many one-inch cubes have paint on no side?

Test 5. Geometrical Objects Test. (Time: 90 sec.)

Form a mental picture of each object, and count the number of straight lines which it would take to construct each one in space.

1. A wedge.
2. Four triangles attached to a square, bases coinciding with the sides of the square.
3. A square box with a lid attached. Assume lid as open and having no thickness.
4. Three squares attached to an equilateral triangle, one side of each square coinciding with a side of the triangle.
5. A triangular pyramid resting on a triangular prism, bases coinciding.
6. State the number of *bounding lines only* in a window with a triangular top and a square bottom.
7. A wedge resting on a cube, edges coinciding.
8. State the least number of lines necessary to draw a square window divided into four panes.

Test 6. The Word "Material" Test. (Time: 300 sec.)

Form as many separate words as you can, using the letters in the word "material." Write words as they are formed on a slip of paper and turn under each word as soon as it is written.

Rugg continues:

Ibid., p. 39: Tests 1 and 2 are regarded as giving measures of the efficiency of the manipulation of spatial elements of a purely nongeometrical type.

Ibid., p. 41: Test 3 is believed to be a measure of ability in mental manipulation of quasi-geometrical elements (plane figures — no three dimensional objects).

Ibid., p. 42: Tests 4 and 5 are regarded as measures of ability in mental manipulation of strictly geometrical elements.

Ibid., p. 40: In the solution of Test 6, it is possible to bring to bear — either separately or in combination — at least five mental functions, *viz.* visual imagery, auditory imagery, systematic method (or "system"), reorganization of the letters previously used, and ideational or perceptual devices.

The course in descriptive geometry was the regular course given in the freshman or sophomore year to all engineering students at the University of Illinois.

Concerning this course Rugg says:

Ibid., p. 34: Its fundamental purpose is to build up for the student a method of picturing solid objects (*e.g.* machines, like engines; dynamos, their constituent parts; and the diverse materials used in engineering construction of all sorts) on one plane, called the plane of projection, and coincident with the drawing paper. . . . Since the course is regarded, primarily, as a "disciplinary" course, each successive problem in the course is designed (among other things) to function as a definite bit of discipline in spatial picturing.

In the strictly geometrical field (Tests 4 and 5) the students in the descriptive geometry classes gained 31 per cent more than the others, in the quasi-geometrical field (Test 3) they gained 20 per cent more, and in the nongeometrical field (Tests 1, 2, and 6) they gained 7 per cent more.

For a complete description of this experiment, one must go to the original memoir. On page 97 Rugg remarks:

Translated into terms of transfer, what do the above facts indicate? They indicate, at least to the author's satisfaction, that the training in mental manipulation of a geometrical character received by students in descriptive geometry operate (1) so as substantially to increase the student's ability in solving manipulation problems of a geometrical nature, but entirely unrelated to the content of descriptive geometry itself, (2) so as to increase the student's ability in solving manipulation problems of a slightly geometrical nature, . . . (3) so as to increase the student's ability in solving problems in mental manipulation of a purely nongeometrical nature, *i.e.* those in which the point, line, and plane do not appear in any way whatsoever.

In commenting on the possible agencies of transfer, Rugg remarks (page 115 of the same memoir):

> It is concluded from this study that the agencies of transfer are the following:
>
> 1. Many specific adjustments or reactions to familiar cues of visualization have tied together training and test series and have undoubtedly acted as bonds or agencies through which more efficient response is made to a given situation.
>
> 2. An important phase of the successful response to a new situation is the building up of attitudes of orientation in the general visual field.
>
> 3. Practice in extending the range of attention has given increased facility in holding and manipulating a large number of visual elements at the same time.
>
> 4. The study of the effect of training on various grades of scholastic ability pointed to the probable effectiveness of conceptualizing abilities in developing methods of analysis and attack. The introspective material confirms the view that these conceptualizing abilities play an important part in determing the method of transfer.

In comment on "The Possibility of Disciplinary Outcomes in School Studies," Rugg finally says (page 116 of same memoir):

> The possibility of one "disciplinary outcome" in a specific school subject, *i.e.* ability in mental manipulation of spatial elements, has been established in this investigation. The writer believes that formal school subjects find a large part of this disciplinary value in the developing of this ability to analyze the problem and to organize a method of procedure; to build up ideals, or to organize method of attack. But it is undoubted that they also make habitual, or automatic, many specific constituents of the complex abilities that function in many complex situations. The successful habitualizing of these specific reactions is accentuated by the building up of a background of fundamental attitudes or orientation, or familiarity with the content of the situations to be met.

15. Miscellaneous opinions. — Even those who are most frequently quoted in opposition to the doctrine of formal discipline sometimes express opinions which imply their belief in a very wide and important carrying over of the results of training. In McLellan and Dewey's *Psychology of Numbers*,[1] page 2, we read:

> In the first place, the study of psychology has a high *disciplinary* value for the teacher. It develops the power of connected thinking and trains to logical habits of mind. These qualities, essential though they are in thorough teaching, there is a tendency to undervalue in educational methods of the present time when so much is made of the accumulation of facts and so little of their organization.

What a wide influence on life and thought is here imputed to the study *psychology!* No special logics for this and for that, nor one kind of thinking for this and another for that, but "power of connected thinking and . . . logical habits of mind [in general?]."

Again on page 3 of the same book we read:

> It will be readily admitted that this tendency to exalt facts unduly may be checked by the study of psychology. Here, in a comparatively abstract science, there must be reflection, abstraction, and generalization. In nature study we gather the facts, and we may reflect upon the facts; in mind study we must reflect in order to get the facts.
>
> This analytic and relating power, which is an essential mark of the clear thinker, is the prime qualification of the clear teacher.

Commenting on a table showing among other things a high correlation between ability in multiplication and division, Brown and Coffman remark:[2] "This table also lends weight to the hypothesis that the increasing kinship as shown by the correlation is due to the increase in the amount of *reasoning* involved." (The italics are mine.)

This seems to imply that there is a reasoning ability at least sufficiently general to cover both multiplication and division. When one reflects more closely on the difference in "reasoning" involved in these two operations, one must come to the conclusion that Brown and Coffman believe in a fairly general "faculty of reasoning," rather than in particular, narrow types of reasoning.

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[1] *Psychology of Numbers;* D. Appleton and Company, 1898.

[2] *How to Teach Arithmetic;* Row, Peterson and Company, 1914, p. 30.

In a résumé of the reasons for transfer given by the various investigators, Rugg[1] says:

> The investigators may be grouped in two schools: (1) those who believe that the effect of training is quite specific and who oppose the view that transfer can be possible through any form of "generalization"; (2) those who believe that the effect of practice can be generalized. Numerically, the latter are much the stronger. . . .
>
> Out of the nineteen investigators who contribute to the discussion of the method of transfer, fifteen take the position that transfer is possible through certain factors of generalization. These factors of generalization may be grouped as (1) ideational factors, *e.g.* abilities in the organization of methods of procedure, conceptualizing abilities, the development of improved methods of learning, etc., (2) attention factors, *e.g.* improved methods of distributing attention, of concentrating, of extending the range of attention, (3) attitudinal factors, *e.g.* the development of attitudes of (*a*) orientation in general fields of reaction, (*b*) adjustment to experimental conditions, (*c*) confidence in success, etc. Thus, the burden of the argument is to the effect that transfer is possible through the generalization of various "central" functions.

In a final statement about the general outcome of all the experimental investigations on the problem of formal discipline, Rugg[2] says:

> In résumé, let us list the fundamental outcomes of the experimental literature of mental discipline:
>
> 1. Prior to 1890 no empirical study of the problem of mental discipline had been made, and up to 1900 only three studies were published. Thus, the experimentation in mental discipline is practically a matter of the past sixteen years.
>
> 2. The studies have been primarily of "peripheral" functions, as (*a*) studies dealing with memory abilities, (*b*) studies dealing with sensory, perceptual data or motor-habit-formation, (*c*) studies dealing with specific school activities. These last are few in number and of recent date.
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> 3. The method of study has been principally a laboratory method. Little attempt has been made to study transfer through schoolroom investigations of those "real" abilities most utilized in the activity of our

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[1] *Op. cit.* (p. 3), pp. 20–22.

[2] *Op. cit.* (p. 3), pp. 116, 117.

mental life. Thus, these investigations are subject to criticism on the ground of the purely experimental conditions under which they were conducted, and generalization from them to actual mental situations of life is unsafe.

4. The subjects have generally been adults, selected for their psychological training, and, as a rule, very few in number; to use fewer than six subjects has been typical practice. With the more recent attempts to make schoolroom investigation, school children have been selected as subjects; their number as a rule has been about 20, and the abilities studied have been connected with school activities.

5. But few experimenters have secured definite and detailed introspections to aid them in determining the agencies of transfer.

6. The tests, like the functions studied, may be characterized as largely unreal, unique, and not closely related to the problem-situations of daily mental life. Like the conditions of investigations, they are "experimental" in character. It should be said that they probably do measure rather well the functions under examination. Of the abilities selected for testing, almost none have been studied by more than one investigator.

7. The statistical results indicate almost unanimously that training does transfer. Within the limits of the investigations, the amount of transfer does not seem very pronounced. Judging from the best of the published investigations, it seems possible that with more extended experiments a more decided amount of spread would be found. With a greater emphasis on the central functions in our selection of abilities for experimentation, we may well expect a larger degree of transferred improvement.

8. By one school of specialists, training has been regarded as specific in effect, and transfer has been explained as due to "identical ability-conditioning factors." However, the typical attitude taken to-day is that practice may be generalized and transferred through such factors as: (1) ideational factors, (2) attention factors, (3) attitudinal factors. Thus, transfer is possible with central functions through the generalization of various ones of these factors. The emphasis here is on making the method of learning a conscious matter, the conscious organization of methods of procedure, the conscious utilization of methods of improvement, better understanding of how to use mental tools, rather than any transferable change (through practice) in the constitution of the organism itself.

As indicating a change in the attitude of the experimenters on this general problem, the following remark by Whipple is significant:[1]

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[1] *Op. cit.* (p. 3).

Not long ago, as a result of the earlier experimental studies, it was felt by many that transfer of training was present either not at all or at least in such slight amounts as to be negligible. More recently, the pendulum has certainly swung in the other direction. Experimentation has been directed less toward searching for the existence of transfer than to searching for the kind of transfer present and the conditions under which it appeared.

The following investigation by Starch[1] illustrates two points which should be borne in mind in studying all such investigations. The object was "to measure the effect of improvement in mental multiplication of three-place numbers by a one-place number, doing 50 problems a day for 14 days, upon other types of arithmetical processes." The average gain in adding fractions, adding three-place numbers, subtracting numbers, multiplying four-place numbers, multiplying two-place numbers, dividing three-place numbers, made by the trained persons over the gain by the untrained persons, was 29 per cent. "The average gain made by the trained groups in the practice series, the first day being compared with the 14th day, was 112 per cent. Hence, the gain transferred to the allied arithmetic operations was only 26 per cent of the gain in the practice series itself." Now the gain reported for the control group (who did not take the training in multiplication) in the processes of multiplication was 20 per cent. Hence it is reasonable to assume that those who practiced in multiplying three-figure numbers gained that much, not from the practice but for the same reasons that the control group gained its 20 per cent. Hence the gain in the practiced operation due to the "practice" was not 112 per cent but 92 per cent, and the transfer was 31.5 per cent of this practice gain.

The second point is that the transfer is to *many* different operations. In this investigation transfer to six different operations was studied. Obviously transfer to many other operations, such as subtracting, multiplying, and dividing fractions, might have been investigated. Clearly the *sum* of these transfers would have been very great (the residual gain in adding fractions was 28 per cent). In estimating the value of transfer the large numbers of different operations to which there is transfer must be kept in mind. A five per cent transfer to twenty different operations is as valuable as a 100 per cent transfer to one operation.

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[1] *Op. cit.* (p. 3), pp. 222–223.

16. Fatigue and industry. The one fact, perhaps most generally known in the whole civilized world, at least to adults, is that work produces fatigue. To continue work in spite of a feeling of fatigue is disagreeable and even painful, and for this reason a person very seldom works to the limit of his ability. There is, moreover, a disinclination to work at high speed or under high pressure as to accuracy and effectiveness, which normally keeps one far below his maximum efficiency for the time he actually works.

The economists describe the situation as follows: There is an element of disagreeableness connected with labor, at least at times, which is called the "irksomeness" or "disutility" of labor. The work period usually begins with a preponderance of disutility. The laborer would rather linger, read a paper, engage in conversation, do something else, or enjoy aimless freedom. But for reasons sufficient to overcome this disinclination, he begins work. Rather quickly interest in the work drives out competing desires. The joy of purposeful activity creates a balance of positive "utility" which rises to a maximum, and as the work period wears on (if it is of considerable length) diminishes again to zero and changes to disutility, which increases until the reward for continued exertion is not sufficient to make the worker continue his labor.

This general description is applicable to intellectual as well as physical labor. There is often disinclination to start work which is overcome and changed into *satisfyingness* as the work proceeds and is then gradually changed to a feeling of fatigue or disinclination to continue, which finally causes cessation of work. The question as to whether a feeling of fatigue implies actual decrease in ability to continue work has been studied quite extensively in the psychological laboratories. To quote Thorndike:[1]

> The results quoted give a fair sampling of the amounts of difference found between the efficiency of a function after long exercise of it at the individual's supposed maximum exertion and its efficiency after full rest. These differences are in general very slight. A man can work for several hours at his utmost and at the end do nearly as well as he will after full rest. . . .
>
> It may be that more complex, more creative, and more important functions — such as writing poetry, deciding a nation's policy, or developing a scientific theory — will be found to suffer more from lack of rest than the more mechanical operations that have been measured.
>
> *Ibid.*, p. 41: I doubt, however, whether complexity, originality, and importance in and of themselves make a function fatigue much more rapidly.
>
> *Ibid.*, p. 68: The essential empirical facts about the curve of mental work seem then to be as follows: Two hours or less of continuous exercise of a function at maximum efficiency, so far as the worker can make it so, produce a temporary negative effect, curable by rest, of not over ten per cent, and in most functions still less than that. . . .
>
> The work grows much less satisfying or much more unbearable but not much less effective. The commonest instinctive response to the intolerability of mental work is to stop it altogether. When, as under the conditions of the experiments, this response is not allowed, habit leads us to continue work at our standard of speed and accuracy.
>
> *Ibid.*, p. 76: The curve of satisfyingness need not follow the curve of achievement. The slight amount of the decrease in efficiency, due to continued exercise of a function under the condition that one shall do his utmost throughout, may be accompanied by a very great decrease in the satisfyingness of the process. . . .
>
> *Ibid.*, p. 106: The individuals who felt most fatigued, judging from their reports, and those who felt least fatigued did not differ appreciably in the actual loss of efficiency. . . .

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[1] *Op. cit.* (p. 21), Vol. III, pp. 39 ff.

Ibid., p. 107: The feelings of fatigue, from what little is known of them, thus seem a very poor symptom of the loss of ability. . . . They operate to attract one to work, to stop one from working, and to prevent one from renewing mental work, but they do not tell how badly one will work if he does continue as efficiently as he is able. . . .

Ibid., p. 115: Fatigue in the vague popular sense means that we are less willing rather than that we are less able, that the probability of achievement is decreased by the increased effort that it requires, rather than that the possibility of achievement is decreased inevitably.

If we are to give full credence to these statements by Thorndike, and there seems to be very good reason why we should do so, it may turn out that becoming attracted by intellectual activity on the one hand and on the other hand becoming inured to its dissatisfyingness, when long continued, are results of education which are of the greatest value; provided, of course, that such results can be attained. The development of habits of intellectual industriousness may be of as great value as information and ability. In our treatment of the main topics of this book we shall frequently have occasion to refer to this section and the one following.

17. Overcoming fatigue — incentives to industry. — It is no occasion for surprise that continued application to uneventful mental labor should result in a feeling of dissatisfaction which we call *fatigue*. In the long course of human evolution, there has been little to prepare us for long-continued direction of the attention into one narrow channel as there has been little to prepare us for long-continued physical labor of the kind we now do in factories or mines, on the farm, or in the shop.[1] To make matters worse, our own troubles always loom greater than those of others, and the work in which we are engaged seems more dissatisfying than other work. So it comes about that we have a constant tendency to quit what we are doing (when it becomes irksome) and do something else. Other work is never so attractive, the great outdoors is never so inviting as when we are forcibly kept from them by our appointed tasks. To quote Thorndike[1] again:

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[1] The writer once knew a man so "lazy" that his wife had to cut wood for the stove, though he would gladly tramp 20 to 30 miles a day following dogs and a fox. Such men are not rare; they are just a little nearer our far-off ancestors than most of us. It was not physical exertion that annoyed this man but the prosaic, orderly activity which we call labor.

> It is the fact that we are fatigued by what we do not do as truly, and perhaps as much, as by what we do. For children *not* to run and jump and squirm and sing and laugh and talk is the essence of mental work. For us all *not* to indulge in our favorite occupations is, as hour after hour of reading legal reports, or adding columns, or whatever the task may be progresses, a more and more impressive feature of the task.

But we are not entirely helpless in our slavery; we can learn to like it. The knowledge that what we are doing is a necessary part of a worthy task in which we are greatly interested, the allurement of expected gain or prestige, the exhilaration that comes from successful performance, these are our fortifiers and our saviors. It is well known from laboratory and from workroom that fatigue comes much sooner when the work itself is distasteful and that work which is of real interest enables one to keep on for long periods without painful fatigue. We continue to quote from the same volume by Thorndike.

> P. 120: But it is a fact easily verifiable that interest *does* add to and that repugnance *does* subtract from the amount of work done.
>
> *Ibid.*, p. 122: The extreme condition where the mind seems literally nauseated — will not have anything to do with the problem — may be cured . . . by an increase in the value of the answers to be got.
>
> *Ibid.*, p. 127: The resistance which blocks mental work may be diminished by supplying interest and motive.
>
> *Ibid.*, p. 128: In the case of wise and experienced adults, it is often hard to thus dissolve fatigue by adding interest and motive; they perhaps have already themselves used all available ones. But the rank and file have not thus exhausted the prevention of repugnance and distraction; and children have hardly learned to use any of them.

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[1] *Op. cit.* (p. 21), Vol. III, p. 123.

Ibid., p. 127: The same work done with interest does not so soon produce *ennui* and repugnance. The denial of certain satisfiers, such as games, conversation, or reverie, may be balanced by the addition of new ones, such as money reward, zeal to improve, or confidence that the work will profit oneself and others. The inventor, man of science, or poet, working a score of hours without rest at full efficiency, is not an exception to the laws of work but an illustration of them. The limit of work for every man is elastic at the pull of interest and personal profit.

Perhaps the greatest motivating force of adult life is what President Eliot calls a "life motive." Next to the possession of such a motive the greatest boon to him whose life is destined to much intellectual labor may well be to experience early in life full use of his intellectual powers and thus learn something of his own capacity and the actual meaning of high endeavor. Men who have taken honors in the English universities have said that one great value of their work was that after it nothing seemed really hard. They were permanently inured to the onerousness of mental tasks.

18. Enterprise. — Even in the simplest occupations people differ greatly in their ability to find appropriate means to reach desired ends. One person must be shown in detail each minor act that goes to make up a process; another needs only to know the purpose to be attained and the more general plan of operation, and he may be left safely to fill in minor details as they are needed. There are all degrees of variation, from the most helpless servant who performs a few routine duties with little change from day to day to the man who guides the destinies of an empire, writes an epic or an opera, makes fundamental discoveries, or handles a complicated industrial organization. Some persons can not work without a boss, others work best when they direct themselves; some are utterly lacking in enterprise, while others are highly endowed in this respect.

Can enterprise be developed, or is it a quality conferred *only* through inheritance? There are no laboratory experiments bearing on this point; but it would be very strange, if acting under minute orders for a long period of years and especially during the early years of rapid habit formation should not tend to prevent the development of initiative and self-direction.

On the other hand it would be equally strange, if early and long practice in discovering the best way to do things and in directing one's own activity in doing them in the way thus decided upon should not tend to develop the qualities which we summarize under the word "enterprise."

The remainder of this chapter will be devoted to a brief and informal consideration of the relation of the intellectual tasks performed in the schools to the developing of the very important elements of industry and enterprise.

19. Popular inferences from modern criticism and experimentation. — We shall first consider certain recent trends in educational policy: The practical effects of the early experiments on the transfer of the results of training have no doubt been much greater than were warranted by these experiments and far beyond what would be counselled by the experimenters themselves. When word went abroad that the doctrine of formal discipline had been smashed, the general workers in the educational field did not wait for the formulation of a new psychological theory but drew inferences at once in terms of educational doctrine. They could not afford to be "behind the times." During the last twenty years the writer has made note of practical, concrete conclusions drawn by teachers, principals, and superintendents all the way from the Atlantic to the Pacific: "The performance of a difficult and trying mental task is of educational value only in so far as it furnishes information and acquaints the performers with the details of their tasks and thus enables them to perform more easily strictly similar tasks."

"There is no mental training in any course except for work like that done in the course." "There is no such thing as developing general mental ability by education." "There is no virtue in hard mental labor; the idea that mental power is increased through effort however persistent is dead."

What can be expected of the nonexpert in education when leaders make such statements as these?

> That *doctrine*, therefore, which calls for the establishment of generalized habits is a psychological absurdity. Habits are specific responses to specific stimuli. . . . This means that we must teach each fact or theory worth teaching as if the salvation of the intellectual world depended upon it, for it may be that the limited training one gets from any one of them will fail to modify us in some [any] other desirable way.[1]

The general result has been that the idea of mental exercise for the sake of the exercise has been largely abandoned. One phase of the analogy with physical exercise has gone by the board. As practicing dancing or walking the tightrope or driving an airplane or boxing or playing the piano makes one skillful in performing these particular acts, so does adding numbers or solving problems in algebra or performing surgical operations make one skillful in each of these. On the other hand that part of the analogy has been abandoned which relates to the development of a generally good physique — the strengthening of heart and lungs and digestive organs which serve equally well in all kinds of bodily effort and which are equally valuable for dancing, walking the tightrope, driving an airplane, boxing, or playing the piano. "If you want to learn to do any particular thing with your mind, learn to do that thing." The idea that a generally effective mind may be developed — that there are qualities of the mind which are brought into play in all arduous mental work and which should be developed by suitable exercise and right mental living — is "obsolete."

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[1] Brown and Coffman, *op. cit.* (p. 31), pp. 25–26.

Effort is an essential element of effective bodily exercise. It is agreed that one can not develop the strongest and healthiest body that is made possible by one's inheritance without great and sometimes exhausting effort. The young boy who runs till he is "dead tired," just for the fun of it, bears witness to the preserving and developing instincts which nature has placed within him. In this respect the educational analogy has been abandoned entirely in some quarters. There are frequent sneers at the old doctrine of the value of effort as an agency in developing the mind.

20. The curriculum of easy tasks. — It is an obvious corollary of the doctrine just stated that the hard task has no place in the school curriculum because it is hard — that as far as possible the task should be rendered easy and attractive to those who hate effort and hard work.[1] The changes which have come into the school curricula, even within the memory of those who are now in middle life, in the way of making each particular task easier is nowhere better illustrated than in arithmetic. One needs only to compare the old Ray's Higher Arithmetic, once so popular in our country schools, with the so-called advanced arithmetics now studied in the seventh and eighth grades, to get a measure of the distance we have gone in three decades.

It is not contended that the school curriculum of to-day requires less total work than that of thirty years ago. The modern curriculum has been so "enriched" that the multitude of subjects and topics to be mastered certainly requires as many hours of work and possibly more. To many students the new curriculum is no doubt more tedious and fatiguing than the old. The stones to be placed in the education wall by the student of a half a century ago were of very uneven size. Some were so large that many pupils failed to lift them at all; even the very strongest had to do their utmost to put them in place. The modern wall is built of smaller stones. All but the very weakest can handle them with ease and even the weakest, if they do not border too closely on the moron type, can handle them with reasonable effort. But the number of stones to put in place has been vastly increased.

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[1] There is, of course, no suggestion that a course should be made more difficult than is necessary to understand the subject. Every effort to simplify and clarify presentation is laudable. What is objected to is the dropping out of topics which are fairly difficult but which the pupil can master by proper application and the general avoidance of subjects which require aggressive and somewhat prolonged effort.

"The 'catch' problem is now in disrepute, the wise teacher feeling by a sort of intuition that to willfully require a pupil to reason to a result sharply contrary to that to which previous habits lead him is risky." E. L. Thorndike, *The Psychology of Arithmetic;* The Macmillan Company, 1922, p. 21.

21. Educating democracy. — One of the most far-reaching slogans of the past century has been: "Democracy must be educated." Now "democracy" includes all members of the *genus homo* who have reached a certain age, with the possible exception of some individuals so far below normal as to be classed as morons or lower. It is known, of course, that, though all members of a democracy may be equal before the law, they are not equal either physically or intellectually. Indeed, the intellectual inequality of those who constitute democracy may not be appreciated as generally as it should be. It is a maxim in biology that the least variation occurs in those organs which are of the longest standing in the history of the race, or what amounts to the same thing: The greatest variation occurs in organs which have been developed comparatively recently or which have undergone great recent change. It seems fairly certain that the more complex organization of the human nervous system and the attendant functioning of the human mind are of very recent origin when compared to those bodily structures which we possess in common with the lower animals. Hence, it seems *a priori* probable that there is greater variation in mental aptitudes than there is in the general bodily powers which our ancestors have possessed during long geological ages. We must bear in mind that this variation extends downward from the average man as well as upward. But our school curricula have been devised with "democracy" in mind. We have provided stones to be placed in the educational wall which are not too heavy for the general run of students and generally not too heavy for the weaker.[1] It is not at all unlikely that our "curricula of easy tasks" owe their origin partly to a desire to provide an education which shall be "within the reach of all," and it may be that the doctrine of "nontransfer of abilities which are developed by education" has been made use of to justify conditions which by itself it could not have brought about. When practical decision is rendered for reasons which would not be attractive if boldly stated, it is not uncommon to seek theoretical justification elsewhere.

22. The "practical" in education. — Education has always been regarded as a preparation for life. In the days of formal discipline it was conceived that the best way to prepare for life was to devote a large part of educational effort to the general upbuilding of the mind. With the abandonment of the doctrine of formal discipline, comes naturally an increased emphasis on direct and immediate preparation for the tasks which it is believed the individual will be called upon to perform. Furthermore, the young pupil must be taught these tasks as nearly as possible in the exact form in which he will be called upon to perform them. In arithmetic, for example, it is insisted that the operations and forms be as nearly like those used in actual business as may be. In cases where several different forms or methods are used, such as in computing interest, it has been urged that all forms in common use be taught in order that the product of the schools may be fully prepared for the duties of life, whatever these may turn out to be. To quote from E. C. Moore:[1]

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[1] "There is at present a bewildering and unsuccessful attempt at comprehensiveness. It fails in its purpose in giving neither adequate information nor discipline. It asks too much of the student and too little." Secretary of State Charles E. Hughes in a paper entitled "Aims in American Education" read before the 1922 meeting in Boston of the American Education Association.

> We can not even say that learning to solve the arithmetic problems of the book will help [not even help!] us to solve those of the market unless our study of arithmetic in school has been of so broad and concrete and practical a kind that we will *inevitably* recognize the applicability of what we learned in school to the problems that now face us and demand solution. There must not only be a common element in the practice work and the life work, it must be a *usable* common element. The only way to make it usable is to make the student keenly aware of its connections with the *very* matters upon which he is to use it.

The italics in this excerpt are mine. The connection must be "*inevitably*" recognizable — inevitably *by whom?* Evidently, even by the weaker student. The "common element" must be "usable"; again, no doubt by the weakest as well as by the strongest; and these ends are to be achieved by making "the student aware of its connection with the *very* matters upon which he is to use it." What glorious possibilities one might look for when a none too critical teacher tries to carry this general doctrine into concrete effect!

There are signs of late, however, that we are receding somewhat from this extreme position. It is now urged, for instance, that one method of computing interest be taught and that that one be taught thoroughly. However, the retreat from this extreme position does not seem to be dominated by any clearly defined doctrine of educational psychology but rather by the pressure of common sense.

23. Early specialization. — It is a direct corollary of the "practical-education" point of view that specialization should begin early. It is quite true that all members of a democracy should know how to read and write and should have some information about civic affairs and about history and geography. A moderate amount of arithmetic will suffice for the average citizen; and when one reflects on the lives of our laborers, our farmers, our men of business, and even professional men — all of these except a very small number indeed — one may be permitted to doubt the extent to which literature, such as is studied in the schools, plays a permanent role in their lives. The thing that really counts in life, aside from discharging the normal duties of citizenship, is the making of a living or a fortune. Whatever we may say *should* be, the fact *is* that, in the case of the vast majority of us, by far the greater part of our effort and thought is directed to the doing of things which will bring us money. In most cases the money may be needed to obtain the necessities of life; in other cases it may be used to accumulate a competence or a fortune. This being true, an education which shall prepare truly for life must prepare for the making of money. (This statement would be less rasping if it read "prepare for one's *life work*" but its meaning would not be essentially different.) Inasmuch as most children leave school at an early age, say sixteen or less, this preparation for the making of money or its equivalent must begin early.

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[1] *Op. cit.* (p. 3), p. 90.

Since life, and especially school life, is too short to prepare thoroughly for all, or even a few, of the vocations, it becomes necessary to begin very early to look forward to a special vocation. Thus, if the boy is to be a farmer, he should begin early to learn about the conditions of crop growth — about the thousand-and-one things which the farmer needs to know. The geography of South America or the history of the Crusades may be interesting (very likely they are not), but they have no direct bearing on the great and abiding problems of the farmer's life.

To avoid waste of time and to insure the best preparation for a vocation, be it that of farmer or doctor, of artisan or banker, it is necessary to select the vocation or profession early and then to shape education with reference to the one selected. Thus runs the logic of the situation; and we are not permitted to escape the necessary consequences of our premises even though they are distasteful.

It is, of course, clear that there are great difficulties in selecting a vocation or profession early in school life. Tastes and inclinations or external conditions may develop later which will make a different selection highly advisable or even imperative. The old doctrine of general mental discipline was much more comfortable in this respect: According to it, it did not matter so much what was studied as how it was studied. Any mental work well done was a good preparation for any vocation, the selection of which could wait. In the meantime the mind, it was thought, was made strong and effective, ready to be turned upon any problem or task which the exigencies of the future might bring. It is a necessary consequence of the extreme modern doctrine that our salvation lies in providing in our schools preparation for all walks of life, in deciding early in life what vocation each child is to follow, and in giving him the best possible preparation for that vocation.

24. The common-sense point of view. — As much as the supposed demolition of the doctrine of formal discipline has influenced the making of school curricula, as often as teachers and school officers have repeated the well-worn remark that the doctrine of formal discipline "was exploded long ago," it is nevertheless true that people in general — both the lay and the learned — retain certain general beliefs, which make one feel that the old doctrine has not altogether been driven completely from people's minds. It is generally believed that a "good education" helps its possessor to conquer new problems as they arise.[1] The writer knows a farmer in Wisconsin who attributes his unusual success as a farmer to the fact that he took a five-year course in a strong university — a course in engineering, as it happened. To quote him: "The nasty problems which I had to solve in my course at the university opened my eyes to the possibility of systematic and persistent attack on any problem. So, when I came to farming, I went into the thing thoroughly. I knew what it meant to endeavor to exhaust the possibilities of a situation. The result has been that unlike most farmers I have been learning something every day. My most valuable assets have been an abiding skepticism that I do not know all about my work and a faith that I can learn. These qualities I attribute largely to my work in the university and especially to one year of graduate work in which my principal work was, by the way, the testing out of different types of electric motors."

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[1] This common-sense view is excellently expressed by Charles E. Hughes in a paper cited on page 44, in which he urges a return to the old time cultural studies: "It [democracy] needs men trained to think, whose mental muscles are hard with toil, who know how to analyze and discriminate. . . . I am one of those who believe in the classical and mathematical training and I do not think we have found any satisfactory substitute for it."

Testimony of this sort may be multiplied *ad infinitum.* It would be difficult to find a man or woman who has accomplished a piece of difficult and trying work who does not feel that this has rendered him or her more capable of accomplishing not only a similar piece of work but a difficult task of any kind whatever. This, it is believed, represents a widespread point of view which may be called the *common-sense point of view.* It may be wrong, as many common-sense points of view have been. We know that the earth is round and that those who live "under" the earth walk with their heads in the direction in which we have our feet and do not fall off the earth, common sense to the contrary notwithstanding.

Nevertheless such a point of view deserves consideration. As Herbert Spencer has remarked, any widespread belief creates a presumption that it has a certain basis in fact though that basis may be very different from what it is supposed to be by those holding the belief.

We therefore proceed to a more detailed statement of this belief and of the reasons urged in its support. In reading the sections that follow, it should be borne clearly in mind that the laboratory experiments which are supposed to have demolished the doctrine of formal discipline have been confined only to the study of ability as defined in section 9, that considerable transfer or spread of ability has been found; and that, moreover, the "abilities" so studied have been of the very simplest kind. The more complex abilities as well as the qualities of "industry" and "enterprise," which are highly important for success, have not been studied in the laboratory and probably can not be so studied with success.[1]

25. Illustrations of personal effectiveness. — A boy of twelve was told to pile some wood. After working a short while he quit and joined his playmates; his grandfather of seventy-five then piled the wood. While actually at work, the boy piled wood faster than the old man, but the grandfather was the more effective worker, because he got the work done. It is related that South American Indians were given oxen with which to plow and that after one day's work they killed the oxen and gorged themselves on the meat. A Japanese farmer works nearly every day of the year for the greater part of his life and never enjoys a feast such as these Indians had. The Indian is superior to the Japanese in stature, in strength, in endurance, in physical ability to handle heavy implements, but the Japanese is incomparably the more effective worker.

In the middle of the nineteenth century a brilliant young Englishman of great scientific attainments was regarded as one of the most promising in a group of unusual men. He was possessed of a fine physique and splendid health, which he retained until his death near the close of the century. His grasp of the fundamental principles of science was a marvel to his companions; his brilliant flights of scientific imagination, his cool and effective discrimination, his wonderful power of lucid expression, all combined to earn for him the expectation of an unusually fruitful life. But from the point of view of actual performance, his life turned out to be a disappointment. He did minor though promising pieces of work in many fields, but never did he do anything which justified his early promise. At about the same time Herbert Spencer, likewise brilliant but never strong in body and badly favored in some other ways, was formulating the outlines of his synthetic philosophy. In 1863 he published a prospectus of a general treatment of biology, ethics, and sociology — in all a set of books consisting of twelve volumes. Then during a period of more than thirty years he labored to complete his stupendous program. Never able to work more than two or three hours a day, at times so ill that he changed the order of his work for the purpose of completing what he regarded as the most essential before death might overtake him, he lived to finish his task, a fit monument to one of the most effective workers that the human race has produced.

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[1] Discussing race differences, Thorndike says: "The nature and amount of race differences in such traits as intellect, enterprise, and self-reliance can not be inferred from the amount of difference found in sensory or sensori-motor traits. . . . So the complex of qualities which is called enterprise remains largely untouched by the psychologist's tests." *Op. cit.* (p. 21), Vol. III, pp. 220, 221.

26. A job or a career. — To further illustrate the difference in effectiveness of two individuals possessed of apparently equal qualifications for accomplishing their tasks, we will relate the story of two men who entered the same bank at about the same time. Each was a graduate of a great university and each was distinguished both as a student and as a man. Both were faithful and absolutely trustworthy. They are still in the same bank — a great institution in the Central West. Twenty-five years ago they held about equal rank and received approximately the same salary. To-day one of these men is one of a number of assistant cashiers. The other is first vice-president of his bank and is in line for the presidency of his own or some other great bank and is receiving four times the salary of the first. When entering upon his career, each of these men got what was considered a "good job." One of them held the job creditably. No one could find fault with his work; he received such promotions from time to time as were natural in the course of events. The other man received a promotion once, and sometimes twice, a year. One of these men had a job, the other a career. The one did his work satisfactorily and let things go at that, while the other had the spirit of the banker from the time he started. His own endurance — not his duties — was the limit of his efforts. He learned a new job each year and learned it well. It soon came to be understood that if a new position was to be filled this man could fill it, perhaps not perfectly at the outset but eventually he would know more about it than any one else. The story of these men goes to the heart of the matter. No man can be educated in the schools so as to make him ready to step into the presidency of a great bank — the education for that position consists in learning the work of a great many subordinate positions and extends over a long period of time. One of these men had essentially "finished his education" when he entered the bank; the other had just begun his. One is an ordinary man, living and enjoying life in a small way; the other is eminently successful. One has taken things comfortably and easily; the other has paid the price of success; he has labored when others were idle; he has denied himself when others did not.[1] Though it is not essential for the present purpose, it may be added that of the two the successful man now looks younger and appears better in every way than does the other. There is a tonic in determined effort crowned by success that nothing else can replace.

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[1] "In general, we have here a point of view from which we may discern a difference between the master and the man of 'all-round' development who is master of nothing. Both have, from the informal experiences of life, some knowledges and skills which fit them to undertake the mastery of a given field. Both have developed their potential instruments of mastery — have 'gone over' the principal items of knowledge and 'gone through' with the principal forms of skill required. The master has not stopped here. He has initiated himself body and soul in the elements, so that after a time such things are to him like letters and words to an educated man. They shoot together easily into new combinations. They are units of mediation, of invention. Meanwhile, to the man who has only a 'good general knowledge of the field,' the feats of the master are impossible and almost incredible. The master's units of thought are each to him a problem. He must give time and pains to each one separately. He can not think with them. He is necessarily a follower, or, if he essays the freedom without the power of the master, he is worse than a follower — a crank." Thorndike, *op. cit.* (p. 21), Vol. II, pp. 101–102.

27. The share of education in developing elements of effectiveness. — The preceding sections may be summed up in the statement that effectiveness consists in a good intellectual machine together with the will and the enterprise to keep it successfully at work. It is granted by all that information may be imparted and specialized skill developed by education. Is it possible likewise to develop by education those other elements of success which have just been enumerated, or are these qualities acquired only through inheritance? It is perfectly clear that the experiments of the demolishers of the formal-discipline doctrine do not touch this question. In this field we are as free to pass common-sense judgments as we ever were, and this freedom the writer proposes to exercise. Nor is this freedom exercised wantonly; we are compelled to exercise it. There are certain situations in which we are forced to act on a basis possessing a low degree of certainty or else stand paralyzed when action is necessary. When a child is lost, we must search for it in the most probable places.

Says *Peer Gynt* to the *Button-Molder* —

But suppose a man never has come to know what Master meant with him.

The Button-Molder —

He must divine it.

Peer —

But how oft are divinings beside the mark —
Then one is carried "ad undas" in middle career.

The Button-Molder —

That is certain, Peer Gynt, in default of divining
The Cloven-hoofed Gentleman finds his best hook.

So we must divine our way, trusting we may be right. The writer will express his divining in the form of an educational faith, for as yet in this field,

> We have but faith: We can not [do not] know;
> For knowledge is of things we see [have seen]
> And yet we trust it comes from thee,
> A beam in darkness; let it grow.

An Educational Credo

1. I believe it possible to develop by education an understanding of the price that must be paid for success — of the essential difficulty of worth-while accomplishments.
2. I believe it possible to develop by education confidence in one's ability to cope with difficulties and the habit of meeting them with a confident, cheerful, and patient persistence.
3. I believe it possible to develop by education skill in directing one's own intellectual effort and sagacity in deciding when the sought-for ends have been achieved.[1]

These qualities "carry over" to all worth-while intellectual tasks, and it is difficult to see how they may be acquired at all in early life, except through education in the schools. Lack of these qualities is one of the reasons for the facts described in section 38.

28. Success and confidence. — Many years ago the writer witnessed at close range the work in a lumber camp. It was a really splendid spectacle to see the great draft horses start the immense loads of logs that were hauled over iced roads. At the signal they leaned quietly into their collars, their great bodies crouched close to the ground, and with a confidence born of many successes they continued to pull with the utmost of their strength as the huge loads started at an almost imperceptible rate. A "green" team just off the farm, though equally powerful and equally willing, were utterly unable to start such a load. They stopped pulling long before the load began to move. Their training had been different from that of the logging teams. On the farm the loads are comparatively light and no effort is ever required comparable to that necessary to start a load of this kind. These horses had to be "educated" especially for their new work. Without training in starting a really heavy load they would, so to speak, live and die in the belief that it was beyond their strength. There is no doubt something similar in the lives of human beings. Failure instead of inciting to further effort is more likely to result in the abandonment of effort.[1] It is a commonplace to say that we like to do that which we can do well and dislike to try to do what we do poorly or to attempt what we can not do at all. So it is that success in solving one problem is likely to lead us to attempt the solution of another problem, and success in solving problems to-day gives us a desire to attempt others to-morrow,[2] while failure to-day may (in the case of those not accustomed to deal with difficult problems is very likely to) give us an aversion which it will require serious effort to overcome.

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[1] It is, of course, clear that this does not state all the results of education nor does it enumerate all the elements of effectiveness.

But the real point to be made here is that a firm belief that we can not solve a certain problem acts as a well-nigh insuperable deterrent from making the attempt even though its solution is of the most vital importance; while a belief that we can solve it makes it very likely that we will attempt to do so if the solution should become important to us. But what kinds of problems do we believe we can solve? Certainly those, at least, that appear to be of no greater difficulty than others which we have already solved. This is the fundamental fact which makes the type of education we receive of such vital importance. If we have been brought up on the "curricula of easy tasks" — if we have never coped with and conquered more complex difficulties — we shall not only find ourselves, when the practical test of life comes, without experience in organizing an attack, but we shall be doomed to utter mediocrity by a belief that all problems save the simplest are beyond our power. If on the other hand we have learned by experience in our school life to unravel more complex situations — if we have learned that the solution of worth-while problems is often a matter of prolonged attention and repeated attempts from diverse angles, that frequently much information must be gathered and organized,[1] and if such efforts have been crowned by success — we shall be possessed of a well-based confidence which will make it likely that intellectually our lives will be more nearly up to the level fixed by our inherited qualities.[2]

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[1] "In the intellectual life the inhibitory effect seems far the commoner of the two. Not getting answers seems as a rule to make us stop trying to get them. The annoying lack of success with a theoretical problem most often makes us desert it for problems to whose solution the existing bonds promise to be more adequate." Thorndike, *op. cit.* (p. 42), p. 278.

[2] It must be added that the problems we are now solving must be sufficiently difficult to fairly challenge our power. Repeating simple tasks indefinitely has little or no attraction for us. (Working on the rock pile in the penitentiary is not an ideal occupation from any point of view.) If at the same time their solution forms a useful part of our general aims, we shall have the optimum conditions for cheerful work.

29. Skill in directing intellectual effort. — The last of the three articles of faith in section 27 requires comment. There are certain elements which all problems have in common. First of all, the details and conditions of the problem must be studied with care. The novice in the solution of problems has no conception of the preliminary work usually required before a solution may even be surmised. Snap judgments, solutions proposed when the nature of the problem itself is not known — these are characteristic of a mind that is crude and untrained. Problems containing elements in common with other problems whose solutions are known are very frequently regarded as identical with these, and entirely erroneous solutions are accepted. This source of grave error is by no means confined to mathematics. The banker who loans money, the merchant who lays in a stock of goods, the physician who diagnoses a case, the lawyer who prepares a brief, the farmer who adjusts his crops to varying prices and to fields of different crop-producing qualities, the teacher who adjusts herself to the peculiarities of a pupil or of a neighborhood — all these, and others, are exposed to a like danger. The trained accountant who is merely an accountant understands that, before he can devise a system of bookkeeping for a certain business, he must know what information the business man wishes to obtain from his books — he understands this *because* he is a trained accountant. The man who really understands the general prerequisites for solving *any* complicated problem whatever knows this without *special training as an accountant*. The preliminary inquiry which he knows is necessary in the study of any problem will necessarily raise all possible questions of this kind. The ordinary accountant, however, if he has been educated merely as an accountant, would probably make a failure of some other task in which the man with proper general training would succeed as a matter of course.[1]

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[1] To avoid a certain kind of rejoinder it may be remarked here (as it is remarked in other places in this book) that the road to this final achievement must be marked by such easy stages as will not discourage the pupil. What is objected to are the vast stretches, now so abundant in our curricula, in which there are no real difficulties to be mastered — nothing to do but labor — and the often unnecessary help that is given when slight difficulties are encountered. Our real task is so to motivate the work that at every stage the pupil will make the best effort of which he is capable.

[2] Consider again the Indian and the Japanese (see section 25) doing farm work. The Indian has the advantage in physical strength and probably in endurance. Their tremendous difference lies in their attitude toward work. There seems to be no reason to believe that this difference is a biological inheritance. The Japanese has not been removed long enough from the aimless life of the savage for natural selection to weed out those who are not industrious; while, on the other hand, I am told that Indian girls, if taken early and put to work, make excellent, painstaking, hard-working nurses. The difference is probably *all* due to education. Now there is no question that the industriousness of the Japanese carries over to any occupation into which he may enter and that the absence of industriousness of the Indian likewise carries over. If it be admitted that the difference in general effectiveness shown by the Indian and Japanese is due to difference in their education (total environmental influence), those who believe in the possibility of any large and general effects of training, have now their case no matter what the details of the argument may be.

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[1] "The great difference, in fact, between that simpler kind of rational thinking which consists in the concrete objects of past experience merely suggesting each other and reasoning distinctly so-called, is this: that whilst the empirical thinking is only reproductive,

In the second place, it must be understood that most practical problems are susceptible of many solutions, no one of which is absolutely wrong and no one absolutely right. Some solutions are simply better than others, and the problem is to select one solution which is as nearly the best as may be. To carry out this program requires patience, much thought and labor. To illustrate: Suppose it is decided that children in the 6th and 7th grades are to be taught only one method of computing interest. Which method shall it be? The untrained person jumps to a conclusion, which may be determined by the method he learned when first studying the subject; it may be the result of a talk with a "practical" business man, who himself may be ignorant of prevailing methods and whose choice of method may have been determined by some incident early in life; or it may be determined by a hurried and partial consideration made at the moment.

The man who knows something about the solution of problems in general will proceed after a very different fashion. Only a meager outline of his procedure may be given here. It will run as follows:

1. A list of all methods that are used at all commonly.
2. A comparison of the difficulty of carrying out each method and its liability to error.
3. A comparison of the naturalness with which the methods may be taught to the child.
4. The compactness of the computations of the various methods and the convenience with which the total computation may be exhibited.

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reasoning is productive. An empirical, or 'rule-of-thumb,' thinker can deduce nothing from data with whose behavior and associates in the concrete he is unfamiliar. But put a reasoner amongst a set of concrete objects which he has neither seen nor heard of before, and with a little time, if he is a good reasoner, he will make such inferences from them as will quite atone for his ignorance. Reasoning helps us out of unprecedented situations — situations for which all our common associative wisdom, all the 'education' which we share in common with the beasts, leaves us without resource." William James, *Psychology, Briefer Course;* Henry Holt and Company, 1892, p. 352.

5. A decision as to which of the above elements is to be given most weight. Is a compact method with little liability to error to be preferred to a more complicated one though the latter may be more readily understood?

In making this study personal habits and predilections must be excluded. It must be remembered that the method to which we are accustomed seems simplest to us; in short, the effort must be made to eliminate all personal idiosyncracies from the investigation.

In the third place, a final checking or testing of all solutions should be made or noted. The checks will be as varied in character as the nature of the problems themselves. In the case of the doctor's diagnosis the final check may be the death or the recovery of the patient; the merchant purchasing a stock of goods has his problem checked by success or failure in making money.

This merest outline of the general method of solving problems (not problems of mathematics only) indicates one of the general results of the study of any subject if the manner of study is properly guided.

30. Qualitatively right — quantitatively wrong. — It is, of course, admitted that the obtaining of one kind of information does not furnish other and different kinds of information; learning the multiplication table does not give information about spelling. It is admitted also, at least for the sake of the argument, that an individual who has learned one operation carries over to the learning of another operation only those elements which are common to the two processes of learning. What is contended is that the elements which are common and which it is claimed are carried over are at least as important as those which are not carried over. It is admitted that there are certain elements which do not carry over; that is, qualitatively the analysis of those who claim to have overthrown the doctrine of formal discipline is correct. When, however, it is inferred that we must abandon entirely the old idea that *general mental discipline* may be acquired by certain studies, the class of nontransferable qualities is assumed to be disproportionately larger and more important than it really is; that is, quantitatively the analysis is believed to be wrong. This is an error which has frequently been made — qualitative accuracy coupled with quantitative inaccuracy. Arsenic is a deadly enough poison, but a trace of it will not kill; it may even be beneficial.

The writer believes that those elements which are common to a wide range of human endeavor are relatively the most important. In devising a course of education for a child these qualities should be kept in view from the outset. By proper encouragement it is not difficult to get a child of seven or eight to begin to plan his own education; to think, for instance, that he should learn to read and to devise his own ways and means for learning to read. It is conceded that the ability to read is not carried over, say into arithmetic; but it is contended that the boy who takes charge, in part, of his own reading and who similarly is led to take charge of his other studies will develop qualities, which will make it more likely that he will properly direct his own mental machinery when facing the affairs of practical life.

The logical conclusions inevitably reached by those who believe that the effects of training are narrowly confined to the immediate field in which it is given — that there are no general by-products of intellectual effort and enterprise — have brought us very near the educational abyss; and it may be that some who have been fed on the "curricula of easy tasks" of the last and the present generation have actually fallen into it. Any one who has become at all intimately acquainted with the aggressiveness with which college students resent the doctrine that hard work prepares for life or is required for success in life (the writer lived in a college fraternity house for a considerable length of time while he was an instructor in a university) is likely to become possessed with the feeling that, if these young people have not already fallen so far that there is no return — and surely too many have joined the commonplace permanently — the return will be made only after a rude and painful awakening and at the cost of much suffering that might have been avoided had their education been properly directed from the start.[1]

31. Initiative — self-direction. — The work in our schools is being done under increasingly close direction and guidance. Supervised study is being extended into the high schools and even into the university. The child and the youth alike are told what to do, when to do it, and how to do it. The direction is increasingly in minutiæ; this thing to be done this half-hour, the other thing the next half-hour, is the program in the grades; and as supervised study invades the high school, we shall have pretty much the same thing there.[2] In the college it is: these problems or these papers for to-morrow and so many more for the next day, and certainly so much to be done by the end of a certain number of weeks, when the test comes. These are all very well. Life in the workaday world has its tasks to be performed in the appointed time. For many these are the only kinds of tasks to be performed. But aside from these minutely directed tasks that come in the day's work, the whole matter depends upon one's own initiative and self-direction. Does it not seem reasonable that eight or twelve or sixteen years devoted exclusively to work under minute and detailed direction and assistance should render a person fit, not for the self-stimulated and self-directed activity of a leader of men but for the directed activity of a subordinate, the one who can work only when he is bossed? And what a tonic and what cheer there is in self-stimulation and self-direction! One can not here escape the outline of the story of the greatest piece of imaginative literature produced by the people living within the present German Republic; and how different it is in spirit from that which soaked the world in blood in the late war!

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[1] "I think, also, that we have done too much to encourage intellectual vagrancy in college." Hughes in paper cited on page 44.

[2] There is no implication here that supervised study is bad *per se;* it may be exactly the reverse of that, and in many cases probably is. What is objected to is indiscriminate and detailed direction (which is very easy to give), rather than stimulus leading to self-direction. If the teacher works with a well-defined purpose to encourage self-direction and leadership on the part of the pupil, she will find a thousand ways to make him take a part in formulating his purposes.

It is in rebellion against the wrong kind of school supervision and direction that Perry cries out: "Can these students ever get back their birthright that they have fooled away? Workshop experience, contact with workmen and nature, must give them some common sense, but it is difficult to see how they can ever get to think for themselves, to act without explicit orders. Can they ever invent? Can they ever become free men?" *Discussion on the Teaching of Mathematics;* Macmillan & Co., Ltd., 1902, p. 11.

The bargain struck between Faust and Mephisto that Mephisto should bring about for Faust a moment so perfect that he should cry out to it,

> "Remain, thou art so beautiful!"

is commonplace enough. The methods used by Mephisto are likewise commonplace: The love affair with the youthful and innocent Gretchen failing, however, to elicit from Faust the desired exclamation; then the Faust of maturer years — a refined, æsthetic Faust—the glorious civilization of Greece conjured up for him; with Helen of Troy his wife, and his son Euphorion; a kingdom in Attica with Faust on the throne; family, beauty, artistic and refined splendor, power — all these, and still Faust did not exclaim to the moment,

> "Remain, thou art so beautiful!"

Now comes that part of the story which particularly interests us. Faust says in effect to Mephisto: "So far you have made the plans and I have followed them. You have said: 'Follow my directions and you will be happy.' Now we shall change rôles. Henceforth I will be the architect of my own life. I will make my own plans and you will be my servant, helping me with your hocus-pocus to carry them out." The change of leadership marks the turning point of the story. In the midst of the arduous and prosaic tasks of draining great swamps and rendering them fit for human habitation, in the midst of this self-imposed labor, acting as a free and not a directed agent, contemplating the far-off results of his labors, Faust cries out:

> "In thinking of these things I could exclaim to the moment,
> 'Remain, thou art so beautiful!'"

Oh, yes, there may be a lesson for us teachers and school officials here. Instead of helping the child and the youth to develop the powers of initiative and self-direction in the only way in which they can possibly be developed; namely, by exercising them, are we not too often playing the part of Mephisto by taking the lead ourselves — the lead over well-approved paths it may be (and so was Mephisto's) but nevertheless the lead? Alas! but too many of those we lead do not have the vital spark that Faust had — they never turn the tables on their Mephistos and assume the leadership themselves; they remain led until the grave. Is there not yet much to be done before our schools will be truly preparatory for the best kind of life?

32. Learn to learn — learn to do. — The constructive part of the present chapter may be summed up in the two sentences which head this section.

Learn to learn as nearly as may be under the conditions of real life.[1] Learn to learn by organizing the subject and doing the work as nearly as may be independently and under self-stimulus. Learn by failure, by repeated trial, and by final success, as all things that are worth while must be learned. It may turn out that this kind of learning is the most practical learning possible, though the subject matter studied may have little or no relation to the life which the learner is to live; and it may turn out that learning facts which by all known standards are practical and necessary — learning these, after the most approved methods of the soft pedagogy of easy tasks — will in the end prove to be the least practical. How old-fashioned! Yes, but why not true?

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[1] "In learning habits it is possible for man to learn the habit of learning." John Dewey, *Human Nature and Conduct;* Henry Holt and Company, 1922, p. 105.

Learn to do under the conditions under which successful men perform their self-appointed tasks. Learn to labor where there is no applause; to be a hero in the secret of thine own closet. Learn to do not only small appointed tasks but also those long and arduous labors which bear fruit a long time hence. There may be more virtue in a playhouse built by a child under his own direction than the committing to memory of the whole spelling-book parceled out by the teacher so many words at a time.

The information which we obtain in the schools matters somewhat. We need to be able to read. Spelling is essential for the compositor, the stenographer, and for those who still use pen and ink to write letters. Some history, some geography, some of this, and of that — these are all well. But after all, what really matters in life is whether we can face a new situation and master it. Have we learned to learn as things in practical life must be learned? Do we have the initiative and sustaining qualities which make us doers of things? These are matters that really count, and the answers to these questions will write success or failure in our book of life.

33. Elements of educational value. — We shall now attempt a partial enumeration of the elements which make a course of study of educational value. In the early part of school life, the information obtained and the skill acquired should be of very general application; and this, indeed, is generally the case. Reading, writing, spelling, some knowledge of physiology and hygiene, of history, civics, and geography, these are of use in all walks of life.

The information acquired should be in the *most general form that can be understood by the pupil.* That is, as far as possible that which is learned should be raised to the conceptual level. To illustrate from arithmetic: A child in the fifth grade knows that 1/2 = 2/4, 3/7 = 9/21, etc., but this knowledge may be of no use in simplifying a complex fraction, such as $\frac{1+\frac{1}{2}\sqrt{3}}{2+\frac{1}{3}\sqrt{2}}$. If, however, the information contained in 1/2 = 2/4 and 3/7 = 9/21 is generalized into the statement,

"*Both terms of a fraction may be multiplied by the same number without changing its value*,"

it will carry over to the problem of simplifying this complex fraction by multiplying its terms by 6. College freshmen studying trigonometry are frequently unable to make this reduction or to understand it readily when pointed out to them because they did not study arithmetic properly in the grades. Similar remarks may be made about the information which should be acquired even in advanced technical courses. Some years ago, I heard an engineer of international reputation address the students of what is probably the best-known school of technology in the United States if not in the world. Among other things he said:

> While here in the school of technology pay little attention to what is called the immediately practical. Devote your best energies to the fundamental mysteries of physics, chemistry, and mathematics. When you get into practical work, you will have little time or energy left for these. If you live to old age, you will have forty years of practice, while the time available for fundamental and far-reaching principles will be very largely confined to your four years here. We are all long on practice and short on theory, and that from Thomas Edison down. The higher we rise in our profession, the more keenly do we feel that we are short on theory.

As far as the maturity of the students permits, the general scope and bearing of each subject should be laid bare. The student should be led not only to know the steps which together constitute the science or the art that he is studying but to understand why these steps came to be taken in the first place and why they are effective in attaining the desired ends. As far as possible the students should be permitted to look behind the scenes. It happens frequently, perhaps usually, that even very good students, who master each component part of a subject, fail utterly in comprehending the subject as a whole. Each separate part of the technique is mastered while the hierarchy of techniques, represented by the whole subject, is not even approached.

Industry in the higher sense should be developed. This means not only willingness to perform each separate and assigned task as it presents itself, though it includes that, but it means also a willingness and eagerness to master the larger task for the sake of the larger achievement. The child studies his lesson for many immediate reasons: for approval of teacher and schoolmates, for grades to be shown with pride to parents and friends, and this is commendable. Later he does larger pieces of work for larger and more permanent personal gains, such as money or distinction or to satisfy his own ideals of a worthy life. The highest type of industry is perhaps that which is dominated by a life motive that receives the approval of one's highest ideals and aspirations. There are all shades of motives for work from that of the tramp who saws wood to obtain food that the immediate craving of hunger may be satisfied to that of the man who works unremittingly and almost without sense of irksomeness on a task which forms a necessary part and parcel of the achievements with which he hopes to adorn his life. At every stage of education the highest possible motive for industry should be invoked. The higher the motive, the more likely it is that the industry will continue. This requires that no opportunity should be lost to give a broad outlook on life and its ideals. Are we to be satisfied with a narrow life devoid of all but the merest intellectual necessities, or are we to be generous in planning our intellectual as well as our material lives? If this question is raised at the earliest moment at which it can be appreciated and raised tactfully but in ever widening scope as increasing maturity makes this profitable, we may hope to lay the foundation for truly worthy lives.

Enterprise should be encouraged at every step. Too frequently an original thought or piece of work is turned down by the teacher with scant attention because it is inferior to the best that has already been done by adults. Pupils should always be encouraged to cast about for ways of their own. There is a great gap between the child in the lower grades who invents a mnemonic device for remembering how to spell a word and the scientist who organizes complicated plans for conquering a difficult problem or the poet who seeks to grip anew the great driving forces of human life, but the achievement of the former is a beginning, a tender shoot, which if encouraged may grow through all the intervening stages and in the end become a magnificent fruitage of human life. Discouraged it may die altogether or become the stunted growth characteristic of him who is fit only to follow where others lead, the servant and not the master.

School studies, then, are of education value in proportion as they contain useful and far-reaching information and as they afford natural means for developing industry and enterprise and for acquiring insight into the technique of intellectual work. The manner of teaching is perhaps of greater importance than the selection of courses, and it is in this belief that the later chapters of this book are written.

REVIEW QUESTIONS

1. State the doctrines of "the mental faculties" and of "formal discipline" in the forms in which they were held in the middle of the last century. Also state the theory of education based upon these doctrines. How was the theory of the "faculties" related to the doctrine of formal discipline? (§§ 2, 3.)

2. How may the doctrine of "mental faculties" be criticized adversely? If this doctrine is abandoned, does it follow that the doctrine of mental discipline must be abandoned entirely? (§ 4.)

3. Describe some of the experiments the results of which have been cited to disprove the doctrine of formal discipline. What important characteristic is common to all these experiments? (§§ 5-8.)

4. Discuss the amount and character of the spread or transfer which these experiments show. (§ 8.)

5. What elements in the results of educational training remain untouched by these experiments? Discuss the importance of these elements. (§§ 10, 12, 18, 27, 30, 32, 33.)

6. What elements are believed to "carry over" to remote fields, (a) according to Angell (pp. 21, 22)? (b) according to Pillsbury (p. 23)? (c) according to Judd (p. 23)? (d) according to Rugg (pp. 29, 30, 32, 33)?

7. Give in substance Thorndike's position in regard to fatigue and efficiency. (§ 16.)

8. Discuss the statement: Mental work of a serious nature tends to inure one to the onerousness of mental labor. On the assumption that this statement is true, discuss its importance. If accepted as true, what effect should this statement have on the character of school courses? (§§ 16, 17, 20.)

9. What has been the effect upon our school courses of the effort to make education popular? (§§ 20, 21.)

10. If it is concluded that the results of learning carry over only to a very limited degree, discuss the result upon educational policy, (a) as to the immediately practical character of the work; (b) as to early specialization; (c) as to the possibility of preparing by educational training for a life that must consist largely in learning new things. (§§ 22, 23, 25.)

11. Discuss the statement: Successful experience in overcoming serious difficulties increases the probability that one will again undertake the solution of difficult problems. (§ 28.)

12. Discuss the "Educational Credo" given on page 53.

13. Discuss the value of initiative and self-direction. Does it follow from the doctrine of nontransfer of the results of learning that these qualities can not be developed by education? If we conclude that these qualities may be developed by training, how will this conclusion affect the character of school courses? (§ 31.)

14. In what ways may school direction and supervision tend to hinder the development of self-direction and initiative? Discuss means by which this result may be avoided. (§ 31.)

15. What should be the teacher's attitude toward individual enterprise on the part of the pupils? (§ 31.)

16. How does general public opinion agree with the doctrine of formal discipline? (§ 24.)

17. Using the words "job" and "career" as in section 26, discuss the types of education best adapted to prepare for each. If the extreme doctrine as to nontransfer is true, is it possible to prepare effectively for a career?

18. Discuss the elements of educational value in school courses. (§ 33.)

CHAPTER II

METHODS OF LEARNING AND TEACHING

34. School and life. — We have heard so often that the school should be made natural and practical, that, as nearly as possible, the conditions of life outside the school should be reproduced within it, and we have repeated this formula so frequently that we have almost ceased to be conscious of its possible meanings. It is obvious that life in the school must differ in some respects from life outside it for otherwise the school would be superfluous. Indeed the difference of life in school from life outside it is in a certain sense the measure of the efficiency of the school. Of two young men at twenty, who started life with about equal inherited endowments, one has lived the unquestionably "natural" life by not attending school a single day, while the other is a graduate of a strong university. The difference in the equipment of these men is due solely to the fact that life in the schools is different from life outside them.

The problem of real interest to the teacher would therefore seem to be, not how to make the conditions of the school similar to those outside it, but how to improve on these outside conditions so as to make them more conducive to that intellectual life which it is desired to develop. Indeed, that is *the* problem of the school. Clearly, the very existence of the school proves that, by common consent, the conditions of ordinary life are deemed inadequate for the purposes of education. In what respects, then, does learning in the school differ from learning outside it? Is the school simply an intensified outside life so that learning is done more rapidly or are there methods employed which are regarded as more effective than those by which we acquire "knowledge never learned in schools"? In this chapter we shall describe some of the ways we learn in ordinary life and also some of the methods used in the schools. We shall first study several instances of learning outside the school.

35. Learning farming. — The writer grew up on a farm and thus became intimately acquainted with the process by which a very great majority of our farmers learn their vocation. It consisted first of all in learning by imitation to perform innumerable simple acts: carrying wood and water for the kitchen, throwing feed to chickens and hogs,[1] bringing up the cows from pasture, weeding the garden, picking berries, digging enough new potatoes for a meal, shooing chickens out of the grain, giving the alarm when the cattle broke through the fence and helping to drive them back into the pasture, picking small rocks out of the field, splitting wood, holding grain sacks when they were being filled, helping to feed cattle and horses, driving a horse used in operating a hayfork, harnessing and hitching up a team, milking cows, harrowing, plowing, cultivating, shocking and pitching grain, husking corn, operating a hayrake, mower, seeder, binder, and other machines. These and many other acts were performed more or less in the chronological order in which they have been enumerated.

The nature of each operation was learned by observing how others did it and by imitating them rather than by trial and error and selection of the effective method. The wee tot followed his father when bringing home the cows and years later brought them home "all by himself." From sheer curiosity (not with any intention of learning to plow) he followed the plow innumerable times from one end of the furrow to the other. Thus he came to note how a plow is handled long before he could handle one himself. Seldom, if ever, was he set to do a task that was too difficult. He was not compelled or even permitted to profit by his own errors. Frequently a farm operation could be split into minor operations some of which would be simple enough for the boy. If he could not cut hay, he could scatter it to dry; if he could not cut grain, he could shock it; and if he was not strong enough to pitch hay upon a wagon, he could drive the team or rake up scatterings with a hand rake.

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[1] The complete act of "feeding" chickens and hogs, including as it does the apportioning of feed, came later.

In many operations in which two or more persons coöperated the boy "helped." He assisted in stringing and stretching fence wire and later learned to drive staples. He helped locate places for the posts and dig the holes; and later built an entire fence without help. In the field he husked a few ears of corn into the wagon years before he could operate a wagon alone. He helped in feeding cattle long before he could be trusted to feed properly a single animal for any length of time. He drove the team while another loaded the hay, which was thrown up by the mechanical loader.

The time of the year when each farm operation was performed and the necessary sequence of these operations were learned without any conscious effort, in much the same manner that he learned the order of arrival of morning, noon, and night, or the time of the coming of the robins and the sequence of eggs and young in wild birds' nests. This information was never used, however, since the time for performing each operation was decided by those "higher up" without making any use of the boy's knowledge. There were no "problems to be solved" at least not any that caused "confusion and inhibition of action while the mind could decide on the proper course." There were of course minor adjustments to be made, comparable in difficulty with keeping a car out of the worst ruts when driving over a rough road, or possibly with the problem solved by a cat or a dog which avoids putting its feet into awkward places as it runs. The individual operations were so simple and so familiar that the only real difficulty was one of manual strength and skill. If by any chance an unusual operation was essayed, the directions given were at least adequate. The situation, a favorite with writers on education, in which the attempt to do something interesting discloses the lack of certain skill or knowledge and thus begets an interest in these was entirely absent. In most cases the necessary skill and information for the task in hand had already been acquired and in the exceptional cases there was analysis by someone else whereby the troublesome element was sequestered. In no case can I remember a persistent effort to learn to carry out a task by consciously attacking each of its constituent difficulties separately.

Each act was performed as a complete unit and not as related to other acts which were necessary to make it useful. The motivation which comes from regarding an act as necessary to a valued end in the sense of the adult was almost entirely absent. Proximity in point of time was the greatest element in giving a sense of relation. The prospect of going fishing in the afternoon sustained the boy for hours as he hoed potatoes in the forenoon, but I can not recall that this work was ever made more endurable by the prospect of a good harvest. The discovery of a ptarmigan's nest was of greater moment than learning that one cow gave twice as much milk as another, and the finding of a method by which the yield from a field might be greatly increased would have been as nothing compared with the privilege of accompanying a neighboring hunter on a raid of a foxes' den which he had discovered. There was, however, an early willingness to "help"; a general feeling that things had to be done and a willingness to "do one's part."

Thanks to an early developed and excellent physique, a fairly active mind, an already well-developed "instinct of workmanship," a willingness and even anxiety to do his part, pride in "being a man," a keen appreciation not only of the approval but the positive admiration of others, and obedience to specific parental direction to work and how to do it, this boy at the age of sixteen was an effective farm worker. With what feeling of approval he looked upon a well-plowed field turned by his own hand, and with what infinite satisfaction he listened to remarks by others upon the excellence of his work! In some of these operations there was a real effort to improve by practice; there was an earnest endeavor to "draw a good furrow." It must not be supposed that this boy was actuated by a desire to learn a vocation which would be of future use. Nothing could be farther from the truth, for he did not have the remotest intention of becoming a farmer. His future occupation, as he saw it, lay between the ministry of the gospel and the captaincy of a steamboat. His mother had set her heart on the former, but fate willed that he should fall between the two and become a teacher.

36. Building a play sawmill and making a sailboat. — In making this brief study of how children learn outside the school, the operation to be described in this section contains an important and, it is believed, a typical element. In the neighborhood in which the writer grew up there were several small sawmills all operated by water power. Due to this it had become the fashion for boys to build miniature sawmills. These were of all grades of complexity and excellence but none, of course, of any practical value. There was the little mill race a foot long made by gouging out a piece of wood, or nailing two boards together, with the crudest possible water wheel suspended on two little stakes driven into the ground. This on a small scale was the basis of all these mills. In the more ambitious ones there was a dam, a mill race possibly ten feet long, a better made water wheel, and a crank operating a wooden saw. The writer built many of these mills, from the very crudest to some ranking among the best. His most distinctive achievement was a modification whereby the water wheel was made to swing a hammer so that it hit against a suspended sheet of steel (an old broken circle saw) making a noise that on a still night could be heard a mile away. Handel could not have conducted his "Messiah" with greater satisfaction than the writer experienced as he listened to the noise made by this contraption. Whatever may have been the case with Handel, a large share of the boy's satisfaction arose from the fact that others could hear the noise. Never for a moment would he have considered erecting that contrivance far out in the mountains where it would not arrest attention.

An examination of the process of learning to build such mills reveals an astonishingly small element of problem solving. Mechanically the work involved the use of knife, saw, bit, hammer, and in some cases a plane. The ability to use these instruments was the heritage of the race. Innumerable hacked pieces of boards bore witness to the use of the saw, as did slits, inches deep, in the seats of kitchen chairs. Sounds from the carpenter shop and elsewhere announced at all times of the day that the hammer was being used. So far as I can remember there was no serious effort to improve the technique of using these instruments. There was no measuring or marking. It is quite true that the more ambitious types were not attempted at first. "I am too little to make one like that," was the reason given and the work was postponed in the absolute confidence that time would give ability, which indeed it somehow did. There was very little determination to improve by practice.

From the engineering side the work was equally devoid of serious problematic elements. The designs were all imitations. There were of course minor obvious adjustments. The mill race and the wheel had to be brought into proper relative positions. This was done by shoving the lower end of the race sidewise or by adding to or removing little pieces from the top of the support. The relative sizes of the various parts, within rather wide limits, were of no great consequence. These details are important for they indicate, what so far as I can remember was the fact, that the building of these mills presented no problem that perplexed and aroused no consciousness of specific shortcomings either in technical skill or in ability to make plans. The "variant" of the noise-making hammer was a happy stroke of "genius" and aroused no interest in further experimentation.

An operation which, in the case of the writer, brought into play certain different elements was the making of sailboats. These boats were made of solid logs, shaped with various instruments, such as axe, knife, chisel, etc., and then gouged out. The work was laborious. In the earlier attempts the construction of these boats was purely a matter of imitation. There was always the difficulty that such a boat would not sail in a straight line. This the writer conceived to be due to the fact that the boat was not exactly the same shape on both sides (it was not symmetrical); so he devised a method for securing absolute symmetry. It consisted briefly in first shaping completely one side of the boat and then whittling a little board so as to fit the shape at a certain point. The board thus came to represent a cross section of one half of the boat at this point. The opposite side was then to be trimmed down so as to fit this cross section. If this process were repeated at many points along the length of the boat the result would evidently be a very close approach to symmetry. I just said "the opposite side *was to be* trimmed." That is the correct statement, for the project was never carried out; it was too laborious.

The devising of the method was a real act of reasoning and not merely a use of the trial and error method by which, for instance, rats learn to escape from mazes or dogs to open doors. It was a rational adjustment of means to an end in which the conditions to be met were clearly perceived and the required adjustments decided upon before trying them out to see if they would work. The fact that this boy never actually used this method for making a symmetrical boat would seem to show that devising the method was of greater interest to him than securing a finished product by means of it. Have we not underestimated the interest which children (at least some of them) have in the purely intellectual elements of life?

37. Learning housekeeping. — For the last decade the writer has watched several members of his household grow and learn. One little lady, not quite six, can lay the table for a simple meal. The process of learning has been going on for a year; her older sister was helping and she wanted to help too. She began by placing the napkins; she was shown just where to place them and that was her complete problem. A little later she learned to place the silver; several showings were necessary before she could place each piece properly. In this manner she added item by item until she could perform the complete act. While reflecting on the subject matter of this chapter, I asked her if she could set the table. She said, "Yes, but I can't reach the dishes. When we have cereals I put on cereal dishes, and when we have fruit I put on fruit plates." That showed exactly the "topic" she had learned last. All the preceding topics were taken for granted. They had been learned long ago and had been "reviewed" so frequently that they were now thoroughly familiar.

An older sister of ten can do many fairly complex tasks. The process of learning was in each case closely similar to that just described. In no case was there an attempt to perform a complex task with a disclosure of shortcomings and a special attention to the difficult elements. New elements were learned from time to time and immediately associated with what was already known to form a larger whole. Instead of an eagerness to conquer difficulties there is a surprising tendency to recoil from them with "I don't know how," or "I can't do that," or possibly, "Show me how."

The motive seems to be "a desire to help" or anxiety to "surprise mother"—pretty much the same element that makes us grown-ups delight in making unexpected Christmas presents. They delight in "cleaning house" when mother is out for the afternoon. Once when there was no maid in the house and the dishes from a dinner party were left for the morning, they got up early and very quietly put the kitchen in perfect order before their mother appeared on the scene. Their reward was "mother's being surprised." Approval, praise, affection, and a desire to imitate the grown-ups—these are their springs of action. Their motives are fundamentally different from the motives of the maid who does her work, in the main, because she knows that otherwise she would lose her place, and from those of their mother who knows certain tasks must be done and there is no one else to do them.

38. What adults learn and how they learn it. — It is surprising how few things of a fairly complicated nature are learned by the ordinary adult. It is true that some millions have learned to drive automobiles, and in past years an even larger number learned to ride the bicycle. Farmers now and then learn to operate a new machine; mature women sometimes learn to knit, crochet, and make new kinds of cakes and candy. But the number of such things learned by any one person in the course of his adult life is surprisingly small. The rule is that, if a way is known in which a task can be performed without prohibitive labor, it will be performed that way in spite of the fact that much less laborious ways might easily be found by a person of fair intelligence in case he were to apply himself to that end with earnestness of purpose.

Furthermore, if we look close, we shall find that new skills and new methods are usually not acquired by trial and error — the method of the "forked road"—and *very* seldom by reasoned procedure. Heaven forbid that one should learn to drive an automobile by the method of trial and error! Imitation, following directions either written or oral — that is how the vast majority of adults learn when they learn at all.[1]

Frequently printed directions are insufficient for the learning of comparatively simple tasks. The agent who sells an automobile to one not accustomed to driving gives several lessons and sometimes prolonged instruction to this last addition to the dangers of the road. Instructors visit homes to teach purchasers of sewing machines how to use them. A man who has given his name to one of the best-known makes of sewing machines tells me that one of the chief troubles of his company is that in spite of written and personal instructions they are not able to get the purchasers of machines to use them for more than a fraction of the operations for which they are designed.

Nor must it be assumed that prospect of financial gain will cause the average adult to make any really considerable mental effort. For decades, an army of extension lecturers, county agents, bankers, etc. have made great efforts to get farmers to keep accounts, but the efforts have been almost entirely in vain. It is not that the work is beyond the farmer's mentality. It can be demonstrated to a nicety that for one who is intimately acquainted with the farm it is less difficult to learn to keep a set of very satisfactory cost accounts than to master the first year's work in algebra in the high school. Nor would it require an appreciable amount of time to do the work — ten minutes a day would be more than sufficient for a good-sized farm. I have talked with farmers who say they can see how the information furnished by a set of cost accounts would be of great help to them in planning their farming. They admit that the time they would spend in keeping accounts would in the long run pay them more money per hour than any work they now do, but the task of learning is so forbidding that they do not have the heart to undertake it. Accounting will never come into general use on the farm until farm accounting is taught as a subject in our schools. Adults who really try to solve somewhat complicated practical problems are very rare.

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[1] It is admitted of course that improvement in manual dexterity and a certain smoothening of mental actions which are always taking place do not come about in this way. These are the effects of what school people call "drills."

Finally, not even grave moral responsibility and the most intimate obligations of love are sufficient to arouse the mind from its slumber. Even with adults the immediate pressure of personal prestige, no matter how frivolous and useless its basis, seems to be more effective. The writer confidently ventures the assertion that in American homes there are many more books on whist than on child psychology.

In the last analysis the number of those who are instrumental in introducing new things into the world is small indeed. To mention a few comparatively recent changes on the farm: Harvesting machinery was developed by a few men whose names are now known nationally and even internationally. The spread of its use was slow. It would seem that intelligent Americans who were bending their backs in the arduous labor of binding grain under the scorching heat of the summer sun would rush to welcome a machine which transformed this work to a sedentary occupation, but not so. Endless and discouraging demonstrations; detailed instruction; purchases and use by a few young, exceptionally intelligent, or venturesome, or gullible spirits; the slow and tardy imitation by the many as the superiority of the new over the old was finally forced upon them—such is the story. Sewing machines, washing machines—in fact, all labor-saving devices were introduced in much the same manner. It is an axiom among those who know that any contrivance to be generally salable must require a minimum of mental effort on the part of the user. It must be "foolproof."[1]

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[1] "Naturally in the past no other method of progress than that of uncriticized trial and error could be expected. . . . Consequently there was nothing to stimulate thought.

There are certain small groups of young adults who are exceptions to the rule. An example will illustrate: For several years past the writer has taught "computation for foresters" to young men who have had some experience in the U. S. Forestry Service. They are without the technical education necessary for advancement and come to the School of Forestry to remedy the defect. They have worked in surveying crews, for instance, whose leaders possessed certain skills which these men recognized as the reason for the differences in rank and pay and for the differences in the physical labor they had to perform. These boys come to the school, therefore, purely as a matter of business and with very definite purposes. They expect to derive immediate and very tangible returns for their effort. Their work in the school is characterized by avidity for definite rules, repugnance to explanation, and willingness to imitate complicated and laborious computations. "Never mind why; tell me how to do it," is a not infrequent remark. Doubtless there are many men of this type in the technical industries.

39. Empirical learning. — The types of learning described in the preceding sections are all characterized by the absence of attention to general principles or to the bases upon which these are founded. Attention to general principles would turn one away from the task immediately at hand, would make it unnecessarily complicated, and would delay its completion. "Sufficient unto the task are the troubles thereof," is the natural feeling of us all. We do things by rule of thumb, by obvious simple adjustments, and

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Besides, thinking has never been popular. It is too difficult. So any means of escaping from it has always been welcome. And when as in earlier days; besides the pains natural to originality, the thinker risked his life, new ideas were rarely made secure until the old had been worn out by the corroding effect of time." Edgar James Swift, *Psychology and the Day's Work;* Charles Scribner's Sons, 1918, p. 10.

"The abilities that have originated the advances in civilization have probably been confined to a very few men." Thorndike, *op. cit.* (p. 21), Vol. III, p. 223.

by making use of certain habituated skills. It has been known since before the dawn of history that turning a corner at high speed involves danger of upsetting but it required a Newton to state the First Law of Motion. Those who use the lever in such forms as the cant hook or the crowbar know something about it qualitatively but not quantitatively, and if they should meet it in an unfamiliar form would fail to recognize it. This is true of other mechanical contrivances, such as the inclined plane and the pulley. It is doubtful if the chemistry of foods has received a single contribution from the hundreds of millions of practical housewives who learned their vocation in the school of life. The farmer has learned much in an empirical way about raising grains, feeds, and farm animals but his contributions to what are now regarded as the fundamental principles of soils, plant growth, and feeding of animals have been meagre. One man in a laboratory is more than a match for a million practical farmers when it comes to discovering the real secrets of the farm. The elementary principles of economics are apparently a sealed book to many of our statesmen and our men of affairs who are making extensive preparations to capture markets in which to sell vast quantities of our products, to build up a merchant marine in which to carry our exports, while at the same time they are trying as far as possible to prevent others from selling to us. Even our courts of law, in their long and honorable history, have failed to discover and put into use certain general principles in regard to the fallibility of human testimony which are of the highest importance in the administration of justice and which have been established beyond cavil through experiments in the psychological laboratory.[1]

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[1] "Experiments have proved that, in general, when the average man reports events, or conversations from memory and conscientiously believes that he is telling the truth, about one fourth of his statements are incorrect. .. . Unfortunately, abundant proofs of the waywardness of memory have not altered the practice of trial courts." Swift, *op. cit.* (p. 80), pp. 285, 289.

40. How the school should differ from life. — One of the most stupendous facts about the human race is that it makes such limited use of its intelligence. The vast majority engaged in our arts and industries could greatly increase their efficiency and decrease the physical burden of their work by taking earnest and persistent thought of how to achieve this end. We witness daily farmers, housekeepers, teachers, mechanics, business men, doctors, lawyers, and numerous others, who could easily simplify their tasks and render their environments more propitious if they were to make a serious effort. Who is not conscious of this fact in his own life? It is true that we need more discoveries, more inventions, and more general dissemination of knowledge; but we are in equal if not greater need of a more generous use in our workaday life of the intelligence and information that we already have. This is one (perhaps *the*) respect in which school life should differ from life outside the school. It is important that we impart some information and develop some skills, but it is equally important that we bring our pupils to a keen appreciation of the multitude of problems whose solution is not beyond the powers of average intelligence and the very substantial rewards that await those who solve them.[1] It is perhaps even more important that we should develop, if in any way possible, habits of intellectual endeavor and a decreased sensitiveness to the irksomeness of mental effort.

One can not escape the feeling that, in some quarters at least, there has been a yielding to the pressure to make life in the school similar to the life of the majority of those outside it — a life of much action and little thought. If school is to prepare truly for life, it must somehow prepare for a life of which thinking shall be a relatively large ingredient. But how? Possibly this can best be done by so arranging the work of the school that thinking will be at a high premium.

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[1] Note in particular Chapters XXVI, XXVII.

41. The scientific method. — By the scientific method we mean the method used in the aggressive search for new truth. This method of study and investigation can be described briefly and is of interest here because of the frequent reference made to it in the literature on teaching and also because of its possible usefulness as a schoolroom method. When a situation of a problematic nature presents itself to the scientist, possible solutions may or may not occur to him. If a possible solution occurs, he tests it to see whether or not it actually is a solution. We shall not stop here to describe his methods of testing. If no possible solution occurs, he institutes a systematic search for one. The solution when finally found and tested becomes a definite fact or principle that may be used in the solution of other problems. This method differs from that of the nonscientific person in the careful checking of supposed solutions and in the systematic search for possible solutions. The *conjectures* and *opinions* of the nonscientific become *demonstrated truths* or *well-known fallacies* in the hands of the scientist. The problems which the nonscientist lays aside when the first serious obstacles are encountered become the objects of the scientist's persistent endeavor.

This difference between the professional scientist and the rest of us lies quite as much in opportunity as in personal disposition and in ability. If the method of the scientist were adopted in thoroughgoing fashion by all, the human race would starve to death. The farmer putting in his crops can not stop to conduct investigations in the chemistry of soils. The engineer designing an automobile must work with the kinds of materials already developed and with their known properties. If he stopped to discover a new alloy, the machine might never be designed. A man in any ordinary practical occupation may be ever so able and ever so inclined to use the scientific method but the exigencies of his practical task leaves him free to solve only the simplest problems. This is one of the reasons for the facts stated in the preceding section.[1] (See also section 28.) The occupation of mind or body on one task prevents carrying out other work. The remark by the president of a great corporation to the newly-elected head of a department, "You must lay aside all matters of detail; we pay you to sit here and think," bears testimony to the truth of this statement. Within the limits of his department and within certain other limits which we can not stop to indicate, the manager has been elevated to the rôle of the scientist.

42. The topical method. — The topical method is well known to us all. It is the method by which we were taught in school, and which, in the main, we are following in our instruction. When we teach reading, we seek to confine the attention to the interpreting of groups of characters on the printed page as representing spoken words, to the meanings of these words, and to the enunciation of the printed sentences so as to convey their meanings to the listener. Other matters, though suggested by what is read, are passed over. If a direct question in arithmetic, history, or geography is asked in the matter read, it need not be answered. We read Hamlet's "To be or not to be," representing as well as we can what we conceive to be the thought and feeling of the character, Hamlet; but we do not stop to consider the arguments for or against immortality or the ethical propriety of suicide.

In teaching writing we endeavor to develop skill in tracing out the conventional representations of letters and words without any consideration of the import of that which is written. In my school days no part of the period set aside for writing was used in considering the maxims in the copy books.

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[1] This is indeed the respect in which "life" differs most fundamentally from the school, for in the latter there is leisure to attend to the intellectual phases of things to an extent quite impossible to him who is pressed to complete the tasks of the day's work. We must take care that the school does not surrender this tremendous advantage and thereby debase itself to the level of the learning of the man of the street. Advocates of certain new methods must be on their guard lest they give way to an ever present pressure from certain quarters to reduce school work to mere unskilled labor. There are certain exceptions to the general rule stated in the text such as John Stuart Mill, producing his *Logic* and *Principles of Economics* while employed in a government office, but such exceptions are so rare that the general statement is entirely justified.

In this manner the total of the subject matter which we teach and study in the schools from the lower grades to the advanced graduate work in the university has been divided into distinct subjects, such that each subject is supposed to have a natural logical coherence. The various parts of the same subject are supposed to be permeated by certain principles or processes, such that when a principle or process has been mastered, it can be used throughout the subject or at least in a large part of it. This division into subjects is based on the belief that it facilitates mastery inasmuch as it makes possible the concentration of the attention upon a small number of logically (naturally) related principles or facts, or upon skills having a maximum of common elements without, for the time, distracting it by other objects of thought.[1] This arrangement does not in the least preclude a demand that in the work in any one subject reasonable ability must be shown in other subjects which are necessary for its prosecution. The teacher of history is entirely justified in requiring papers written in respectable English, and the teacher of science is justified in his demand that simple problems in arithmetic shall be solved correctly.

Each of the main subjects is in turn subdivided into topics and these are arranged in sequence with a view to their chronological order of introduction. This sequence of topics and subtopics within each subject is determined partly by logical necessity and partly by considerations of the interests and capacities of the learner. "Is the arrangement logical or psychological?" is a question often asked. In many cases, especially in arithmetic, the logical considerations are imperative, and it is believed that in these cases, in arithmetic at least, the logical and the psychological orders are identical. From the purely logical point of view subtraction *might* conceivably be introduced before addition, but addition must of *necessity* precede multiplication since it is used in carrying out the process of multiplication. Subtraction and multiplication must precede division and the operations on integers must precede the corresponding operations on fractions since the latter are carried out by means of the former.

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[1] "Now in considering each gland . . . as a separate entity . . . we of course commit the usual sin of the intellect: the sin of abstraction and isolation of its material. This crime of analysis the intellect commits every day in its search for truth. Before its dissection it seems to have to dip the elusive article in a fixative, and bottle it in a vacuum." Louis Berman, *The Glands Regulating Personality;* The Macmillan Company, 1921, p. 96.

43. Criticism of the topical method. — The most obvious criticism of the topical method is that it is unnatural — that the attempt to cut across a wide range of human experiences, as is done for example in arithmetic, sequestering out a few strands for intensive examination while taking scant account of the complex matrix in which they are imbedded forces upon human nature tasks for which it is not adapted. The human mind, it is argued, is possessed of precisely those capacities which were necessary in man's struggle for survival. But learning by the topical method is of very recent origin; its age is counted in decades or at most in centuries and this is as nothing in the course of human evolution. Hence, it is argued, we are *a priori* led to expect that solid substantial attainments will not be achieved by this method.

Again it is urged that studying by this method does not prepare effectively for the practical use of that which is learned. The tasks to be performed in practical life are highly complex, be it the preparation of a meal or the building of an ocean steamer. Suppose such a task has been analyzed into many simpler tasks and the doing of them separately has been mastered. It does not follow that the complex task can be performed; indeed it is pretty certain that it can not. The synthesis of many skills into a more comprehensive skill requires learning just as much as each of the simpler skills. For this synthesis, it is said, the topical method does not provide, and its products are as useless as a thousand separate pieces of an automobile would be in the hands of one who does not know how to put them together.

Finally it is urged that the topical method does not motivate the work. It fails to show the use to which the subject is put in the lives of men. It does not create a need which is satisfied by that which the student labors to acquire. An isolated fact is not interesting while in relation to other facts it may be of absorbing interest. The acquisition of a skill may be unspeakably dull and even repugnant if it is put to no interesting use, while the use of it in doing that which is of real interest to the learner may create an enthusiasm which will make its mastery a positive pleasure. For a person of mature years learning to ride a bicycle is, in and for itself, no pleasant undertaking. An awkward physical struggle with apparently little or no progress, falls and bodily hurts, embarrassing situations which make early morning hours and unfrequented spots the favorite time and place — all these must be entered on the debit side. But in the heyday of the bicycle the desire to ride made the learning comparatively easy and millions mastered it with no prompting save their own desire. If we were to devise a fit punishment for the most fiendishly inhuman criminal, we might arrange an endless series of skills comparable to that of riding a bicycle and compel him to learn, or nearly learn, one after the other. As soon as he began to master one, he would be passed on to the next. Such, it may be argued, is the life to which we condemn children who are forced to master skill after skill with little or no opportunity to use them as they are being used in the world about us. They are "forever climbing the climbing wave" or, to change the metaphor, they are forever rolling stones up a mountain but before they reach the top the stones invariably break away and roll to the bottom where the fruitless task must be taken up anew.

44. A rejoinder. — Things are not what they seem and frequently not as bad as they seem. Critics of "things as they are" are seldom judicious or temperate. The windmills which they see in the distance become portentous giants which in their quixotic fervor they charge with fine fury. To comment in order on the points made in the preceding section:

The theory that solid substantial attainments can not be achieved through the topical method does not square with the facts. That is the only method by which substantial achievements in learning and discovery have been attained. Every civilization owes its existence to those who have singled out special subjects for their lifelong and often exclusive attentions. The fundamental secrets of nature embodied in the great sciences of mathematics, physics, chemistry, geology, biology, astronomy, psychology, economics, and sociology have been revealed by those who, contrary to the ways of a primitive world, followed the strands of their own special subjects, refusing to be diverted by the countless other strands of the matrix in which they were imbedded. To use another figure, they followed the scent of a particular phase of truth as the bloodhound follows a single trail though it crosses a thousand others. Indeed as the sciences have become more highly developed each has been subdivided and an investigator has his own narrow special field. Every biologist, for example, learns the general elementary principles of biology as a whole and also of other sciences, but the major part of his life is spent in a small isolated part of his own science.

The general principles developed by the pure scientist have been applied to practical use by others who devoted themselves with equal singleness of purpose to the achievement of special and comparatively isolated ends. To mention a few of these achievements: Agricultural machinery, textile machinery, the steam engine, the machinery and the technique used in extracting metals from ores, the steamship, railway, electric dynamo, telegraph, telephone, phonograph, electric light, moving picture, automobile, airplane — these and countless others were invented and brought to their present state of perfection by specialists. Our commercial institutions, such as banks, wholesale houses, and retail stores, are the products of those who devoted themselves exclusively to their creation. The same is true of those who have created the finer efflorescence of our civilization — its painting, sculpture, architecture, poetry, the drama and the novel, its operas, oratorios, and other music. The list is endless.

Now then, how do we know that that which was learned in the first place by isolating it for separate and prolonged attention and, so far as we know, could not have been learned in any other way may not also be studied most effectively in a similar isolation by those who are not discoverers of new truth? *A priori* the answer "no" is by no means justified. It must be justified, if justified at all, by extensive experimental comparison of the effectiveness of this method of studying with other methods. Needless to say, such comparison has not yet been made or is even in the process of being made. Indeed a rival method, though suggested, has not been put into a form in which its proponents are willing to recommend it for exclusive use. It is true there is some difference of opinion as to what shall constitute a "subject." Some favor "general science" for the first year in the high school, which shall contain a little physiography, a little physics, a little chemistry, and a little biology, but that does not mean the abandoning of the topical method. Even in the memory of men now living we had professors of "science" in well-equipped universities. It is by no means an implication of the topical method that the materials of instruction in the lower schools should be divided into "subjects" in precisely the same way that this division is made in the higher schools. Indeed there is good ground for the opinion that "subjects" in the lower grades should be fewer and more inclusive than in more advanced work.

The assertion that the topical method does not prepare effectively for the practical use of that which is learned must be based on the assumption that this method is not compatible with adequate synthesis in the schools of the separate subjects into the complexes of skill and ability required in practical life. We shall consider these matters further in section 47. For the present we will remark that it is by no means necessary to admit that in all cases such synthesis in the schools is desirable or even possible. There are all gradations between the education which enables its possessor to step into a niche in the industrial world and immediately fulfil its requirements completely and effectively — the most boasted product of the trade school — but which does nothing more, and the education which gives mastery of far-reaching principles that serve in a lifelong career of conquering new difficulties. In the former case education is finished; the complete synthetic skill is provided and the late learner has nothing to do but to use it. In the latter case the synthesis of separate skills and the bringing to bear of different principles upon new situations as they arise are the daily tasks of him whose education continues through life.

But it does not follow that the topical method should be rejected even in the former case or that in the latter case the work of synthesizing skills, principles, and facts should be postponed until after the diploma has been conferred. Far from it! In the most sordidly practical business college where they "fit more completely for business in two years than the public school does in four," the topical method is followed most severely. As the separate subjects are studied there is less correlation and less "motivating work" than in the supposedly less practical schools in which more time is required. In such schools there are final "practice courses" in which the separate threads of the various earlier courses are woven into the complete wholes required in practical work. At the other extreme of our so-called practical education are the great schools of law, medicine, and technology, and in them also we find the same initial study of separate subjects with a final synthesis of these in later years of the course or in post graduate courses. So far, schools of high and low degree are proceeding with a surprising unanimity on the assumption that the topical arrangement is the most practical. Up to the present that is the best light we have. It should also be pointed out that a high degree of isolation of each subject as its study proceeds is by no means inherent in the topical method. As the child goes on with his study of arithmetic, he is also carrying forward several other school subjects and is living a complex life outside the school. These may be, and are in fact, drawn upon heavily for problems and exercises; the word "correlation" has a real meaning. At the same time it is possible to make considerable use of arithmetic in connection with other school subjects, and we encourage the pupil in every way to use it in his outside life.[1] At least, we certainly should do so. The keeping by the child of his own cash account is a case in point.

The last point in the preceding section is considered fully in Thorndike's *The Psychology of Arithmetic.*

> One false inference about the problem-attitude is that the pupil should always understand the aim or issue before beginning to form the bonds which give the method or process that provides the solution. On the contrary, he will often get the process more easily and value it more highly if he is taught what it is *for* gradually while he is learning it. . . .
>
> A second inference — that the pupil should always be taught to care about an issue and crave a process for managing it before beginning to learn the process — is equally false. On the contrary, the best way to become interested in certain issues and the ways of handling them is to learn the process — even to learn it by sheer habituation — and then note what it does for us. . . .

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[1] "Every recitation in every subject gives an opportunity for establishing cross connections between the subject matter of the lesson and the wider and more direct experiences of everyday life." John Dewey, *Democracy and Education;* The Macmillan Company, 1916, p. 191.

A third unwise tendency is to degrade the mere giving of information — to belittle the value of facts acquired in any other way than in the course of deliberate effort by the pupil to relieve a problem or conflict or difficulty. . . . Mere facts not got by the pupil's thinking are often of enormous value. They may stimulate to active thinking just as truly as they may stimulate to the reception of facts. . . .

The last false inference that I shall discuss here is the inference that most of the problems by which arithmetical learning is stimulated had better be external to arithmetic itself — problems about Noah's Ark or Easter Flowers or the Merry-Go-Round or A Trip down the Rhine.

Outside interests should be kept in mind, as has been abundantly illustrated in this volume, but it is folly to neglect the power, even for very young or very stupid children, of the problem "How can I get the right answer?" Children do have intellectual interests. They do like dominoes, checkers, anagrams, and riddles as truly as playing tag, picking flowers, and baking cake. With carefully graded work that is within their powers they like to learn to add, subtract, multiply, and divide with integers, fractions, and decimals, and to work out quantitative relations.[1]

45. The project method. — The term "project method" is new in the literature of education. It belongs to the twentieth century. It is the word of the hour. As yet there is no general agreement as to its exact meaning, but to many that gives it an added charm. Not a few "love Browning" mainly because they do not understand him. We have been told so long and so persistently that there is something wrong with our schools that any plan or method that promises help is like a "shady rock in a weary land." In the case of the project method its very indefiniteness leaves room for hope. But to speak seriously, there is no doubt that to many of the most searching thinkers on educational matters, the project method embodies an idea of real significance. Before attempting to describe this method we shall describe two teaching situations which would unquestionably be regarded as real projects by all who have attempted to define the term.

A project for a fifteen-year-old boy was arranged in coöperation by the boy's father and the teacher of agriculture in the local high school.[1] The boy was given complete charge of about 150 hens with the understanding that he was to familiarize himself with the best method of feeding, weeding out the poor producers, etc. The father furnished the hens to start the project, the feed, and shelter. The boy was to do all the work that could be done by him when not attending school. One third of the cash proceeds went to the boy with the understanding that the money was to be saved to help defray his expenses at college. Under this agreement the boy learned and carried out the complete operation of trap-nesting and learned a good deal about the feeding of chickens. The teacher of agriculture coöperated by answering questions, furnishing literature, and advising on the construction of the facilities for trap-nesting. The boy learned a good deal about the variation in productivity of different hens of the same breed and the importance of getting rid of the poor ones.

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[1] *Op. cit.* (p. 42), pp. 281–284.

In the winter 1895–96 a special property tax was levied in a township where the writer was teaching a rural school. The assessor's figures were obtained and also the total tax to be raised. After computing the tax rate and agreeing with the authorities on the exact rate to be used the seventh and eighth grade classes undertook to "spread" the tax and to deliver to the treasurer a complete tax list. A tax table was prepared, the tax "spread," the work verified, and the list turned over in official form. We did exactly the work which otherwise would have been performed by a public official. To our pride and satisfaction it was announced that there was not a single error in our work.

With these concrete instances in mind we turn to the problem of finding a definition of the "project method." Stevenson[2], who has made a thorough study of the literature, in his book, *The Project Method of Teaching*, says:

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[1] A case which came under the writer's personal observation.

[2] John Alford Stevenson, *The Project Method of Teaching;* The Macmillan Company, 1921, p. 43.

> The definition of the project method which is proposed [by Stevenson] for substantiation is the following:
>
> *A project is a problematic act carried to completion in its natural setting.*

He then proceeds:

> In this definition it is to be noted that: (*a*) there is implied an act carried to completion as over against the passive absorption of information; (*b*) there is insistence upon the problematic situation demanding reasoning rather than merely the memorizing of information; (*c*) by emphasizing the problematic aspect the priority of the problem over the statement of principles is clearly implied; and (*d*) the natural setting of problems as contrasted with an artificial setting is explicitly stated.

We will now consider these four elements following Stevenson in the main: (*a*) Not all physical activities are included under the word "act" as used here. Quoting Dewey, Stevenson defines it as "a series of changes definitely adapted to accomplish an end." "Hence it is opposed to restless and random changes, as well as to mere acquiescence and passive absorption. Dictated exercises, 'busy work', etc., when not accompanied by any sense of a result to which they naturally contribute, are not activity in its genuine, or intellectual, significance; neither is undirected overflow of motor impulse."[1]

We gather from this that physical activities become "acts," in the sense in which the word is used in the definition only when consciously directed toward the achievement of a preconceived end.

On the other hand, "act" as used in this definition is not limited to physical activities.

> There are many different kinds of activity, intellectual, social, religious, and physical. The project does not limit itself to physical activities alone

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[1] Stevenson, *op. cit.* (p. 93), p. 45.

but makes provisions for acts of other types, provided that the individual takes a part in the purpose, choice, and reflection of the directed action.[1]

It is not clear that the individual must take part in all of these ("purpose, choice, and reflection of the directed act"). A conscious direction toward a preconceived end would seem to be the essential element here as in the case of physical activities.

(*b*) It is essential that the project be understood to include a problem; otherwise it could not be differentiated from habits and reflexes. . . .

The project may of course include habits and reflexes [of *course* it *must*] provided, in addition, that there is involved a problem or a situation demanding reasoning for a solution.[2]

(*c*) The problem aspect of the project not only involves reasoning but contains a distinct implication of priority of the problem situation over the statement of principles. . . . The problem [project] method carries with it the implication that principles will be developed as needed and not learned first. Hence the project, which makes specific provision for the problem, lays emphasis on its priority over the statement of principles.

The method of arriving at a solution, where the principles have been learned first and the problem is largely one of applying them, can with greater accuracy be called the original, laboratory exercise, application, or practicum.[3]

(*d*) The problem involved in the project must be carried out in the natural setting which means that the solutions undertaken in the school are [must be?] no different because they are school problems than [*sic*] they would be were they to come up in life outside the school.[4]

It is agreed by many writers that the definition of "natural setting" is beset with difficulties. The illustrations used seem to justify the following: In life outside the school the problem arises in a certain situation out of which it is abstracted for solution. In a project the problem must appear precisely in such a situation and must be recognized as a problem and abstracted for solution by the one who works out the project. In most cases the problems which occur in our texts are identified as problems and abstracted for solution by the one who sets the problem. But this does not cover the whole case as will appear from the following illustration to which we must limit ourselves:

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[1] Stevenson, *op. cit.* (p. 93), p. 45.

[2] Stevenson, *ibid.*, pp. 47, 48.

[3] Stevenson, *ibid.*, pp. 48, 49.

[4] Stevenson, *ibid.*, pp. 49, 50.

> The problem of canning [fruit] may be carried into the field of practice by canning a small amount of fruit in small utensils, and the student may learn the practice of canning or perhaps it is better to say obtain a simple illustration of the practice. But this is not the natural setting of the action. The amount of fruit is unusual. If the student were canning at home, she would have to can a peck or a bushel and would have to use several jars. The process is also unnaturally simple. If she were canning a bushel of fruit, she would probably have simultaneously to watch a fire, get many jars ready, find proper places to set them, or attend to one portion while another is cooking.[8]

The requirement that the student must recognize and abstract the problem for himself is shown in the following:

> "The writer [Stevenson], in teaching the subject of machines to a mixed class, brought to the school yard for purposes of illustration a windlass, block pulleys, and a large board which could be used as an inclined plane. The members of the class were given problems in the manipulation of the pulleys and windlass to show the advantages of the machines. The whole lesson was interesting and probably more spectacular than a demonstration given in the laboratory with model pulleys and windlass. But while this lesson may have been effective, it could not be classed as a project, since the setting was artificial. One of the members of the class, however, had already been busy with the task of lifting baled hay into the barn loft with the aid of a single fixed pulley, which gave him the advantage of direction but nothing else. For him this exercise developed into a project, for he took up the problem which faced him, finding his principles and making the

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[8] Stevenson, *op. cit.* (p. 93), p. 50, quoting Charters.

application, with the result that he installed a pulley into the loft with the ratio 1: 6."[1]

The writer can not resist the temptation to remark that there is something artificial about this 1:6 pulley used to lift hay into the barn loft. For this purpose a horse is always used as motive power and with a single pulley several hundred pounds may be lifted which is quite enough for the purpose of economical operation. With the 1:6 pulley a ton or more could be lifted which would make demands on the structure of the barn and on the placing of the load entirely beyond the possibilities of the situation. The boy evidently wandered into the realm of fiction which is prohibited by the requirement of the "natural setting."

But this is beside the point. We shall close this section by remarking that while the definition given above seems explicit enough, so far as the writer knows, it is not adhered to by any of the writers on the project method. Stevenson himself departs from it repeatedly. He states that "a project is a problem in its natural setting" and this is reaffirmed often enough to make sure it is not merely an inadvertence.[2] It is admitted by all that a problem may involve no new principles while it is stated very explicitly that in the project as distinguished from the problem some new principle should be involved (see Stevenson as quoted on page 94). However, many of the projects described involve no new principles but are introduced specifically to motivate reviews (see page 99).

46. Examples of projects. — It will be instructive at this point to sketch briefly a number of projects that have been reported and to note their purposes. All of those given except the first two are taken from Stevenson who has collected them from many sources. Unfortunately only such as are capable of brief description can be given.

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[1] Stevenson, *op. cit.* (p. 93), p. 51.

[2] Stevenson, *ibid.*, pp. 95, 114, 115, and other places.

1. The project in raising chickens described on page 92 was used to teach a very practical element of farming with no attempt to develop the principles involved beyond what was necessary for carrying out the work.

2. The project of "spreading a tax levy" described on page 93 was used to develop an understanding of how this piece of work is carried out in practice. There was no consideration of cognate matters, such as the purpose of the tax, the justice of a straight property tax, etc. That was no doubt due to the embryonic character of the teacher. Compare sections 171, 179, 203, 209, 210, 214, 216, 219–221, 232–236, and many others in this book.

3. "In one project in physics it was discovered by the class that they were deficient in algebraic equations. It was a third-year class and they had had no algebra since their first year in high school. Lacking the ability to factor and to solve simple algebraic equations, the class made the request that this drill be furnished. The writer [Stevenson] took five hours extra time to complete this project."[1] In this case the purpose of the project was to review a necessary subject in which the pupils were rusty.

4. A class in the second grade found that its game of bean-bag was being retarded because of lack of accuracy and speed in adding. "A project immediately developed out of this situation, having for its aim the development of speed and accuracy in the addition of simple number combinations."

5. Quoting Stimson, Stevenson gives the following as "an illustration" of a "complex project."

> The project of constructing a concrete walk might involve a study of the nature of cement; its action on sand, gravel, and broken stone; its weather-resisting qualities; the seasons in which it might be used; the cost as compared with plank, brick, flagging, and asphalt; the mathematical determination of proportions of sand, cement, and stone to be used; the geometrical determination of the sections into which it should be divided,

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[1] Stevenson, *op. cit.* (p. 93), p. 130.

and whether it should be crowned or flat; the geographical sources of the raw material and the commercial conditions for purchasing the cement."[1]

This is a truly formidable list of topics to hang on the slender thread of building a walk. The purpose of this project is surely sufficiently apparent.

6. In a class in solid geometry it was learned that at the home of one of the members the construction of a driveway was contemplated. "The information desired was: Which side of the house would be the better place for it, and, then, how much excavation and how much material would be required for its construction. The class was given the project and given two days to work out the solution."[2] This is obviously an excellent example of a practical application of the principles already known. (It is explicitly stated that the class *had been* studying mensuration and volumes and it was not a project in the sense that it led to new principles.)

7. Another "project" used in the same class as the preceding was: "The boys needed a number of shots for spring athletics. It was found that a local foundry would cast them for a mere fraction of the cost for which they could be procured at a sporting goods house. However, the foundry needed a model." By collaboration of geometricians, physicists, and those skilled in the manual arts a model was constructed. Its production involved volume of sphere, specific gravity of cast iron, and the turning of a true sphere of a given diameter, and incidentally finding a cube root.[3]

Here we have the bringing to bear of many skills and principles upon a single practical situation, but no "priority of problem" as over against principle. It was the other way round.

8. The study of the proposed widening of Church Street in

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[1] Stevenson, *op. cit.* (p. 93), p. 110.

[2] Stevenson, *ibid.*, p. 256.

[3] Stevenson, *ibid.*, p. 257.

Rockford, Illinois.[1] The project is given as one in mathematics; and in so far as actual school subjects are concerned, it involved arithmetic only. It involved, however, much work in securing information bearing on the cost of the proposed undertaking. The work differed from the traditional problem in arithmetic in that the pupils went to the natural sources (both human and printed) for their information instead of having the information collected by someone else, and also in this that the problem was of real practical interest at the time it was solved.

Many other projects are described by Stevenson. There is the *Better English Week* which consisted in having the members of the class secure first-hand information about the universal demand for good English; slogans were invented and posters made; mistakes were noted and posted. The whole proceeding had somewhat the characteristics of a Liberty Loan drive during the War or of a glorified New Year's resolution.

There is the writing and producing of a Thanksgiving play which leads very naturally to intense activity in many lines.

An antifly campaign is described which led to the study of the life history, habits, and element of danger of the fly and to much practical work in manipulating a social group such as inhabits an American city.

A project for eradicating the barberry bush led to a study of its relation to wheat rust and to much shrewd work in getting bad citizens to be good. Such interesting topics as, "What part has transportation played in developing our country?" "Does the United States produce enough sugar to supply her own need?" have been made the basis for interesting work. The practical installation of electric bells; ventilation; large projects, such as a study of the "construction and furnishing of a bungalow"; the study of the Hebrew people as a type of shepherds under the title: "Study of pastoral people and wool," are titles of projects described. The construction of a fireless cooker serves to make real certain parts of physics. A study of the problem: "What poems written during the War will live?" involves a study of types of poetry. The list of projects already reported is very long and one could invent others *ad libitum.*

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[1] Stevenson, *op. cit.* (p. 93), p. 252.

47. Comments. — It is not the purpose of this brief treatment of the project method to enter into an intensive discussion as to whether or not all teaching situations which are admitted by all properly to be styled "projects" possess the four characteristics described in section 45. To the writer it seems obvious that they do not. It is rather the purpose to give examples of projects of a diverse character and to point out the various purposes for which they are used.

There is one class of projects whose purpose is to teach the doing of a practical piece of work in the best way (at least in a very good way). Such is the project on feeding and trap-nesting chickens. In this case the doing of the job is the complete and final end. It rises to the dignity of a *teaching* project because it points out problems which by the vast majority of raisers of chickens have not been regarded as such[1] and in that there is insistence not only that the work be accomplished but that it be accomplished in the best known way.[2]

Again there are projects which would seem to represent little more than a redivision of subject matter into new topics. The question, "Does the United States produce enough sugar to satisfy her own needs?" can be answered completely by the word, "No," or by giving the amount produced and the amount consumed in a recent year; the information can be found in a newspaper almanac. But it turns out that this is not what is wanted for we find that under this "project" a study is made of the uses of sugar, different kinds of sugar, the places where it is produced, the totals produced, consumed, and imported.[1] Logically "sugar" would seem to be a more appropriate title. "What part has transportation played in developing the country?" is a very natural title for a chapter in a history of the United States. This "project" might well be subdivided into "railways," "steamboats," "highways," "automobiles," etc. Such "projects," or "topics" as I should prefer to call them, have the merit that they group about one center of interest large bodies of information which if distributed piecemeal among other topics might have little significance. Such arrangements are "topical" nevertheless. We must not fall into the error of supposing that the only possible classification into topics is the one to which we have been accustomed. Biographies, the Indians in the United States, immigration, the development of separate industries, etc. are simply topical subdivisions of history with such correlation with geography and other subjects as is always made in any teaching of history worthy of the name. No doubt such work is increasing. Scores of books could and should be written for grade children on such subjects as "wheat," "horses," "highways," "bridges," etc. "The making of a suit of clothes" might be a good title. The making of such books would surely be "projects"; whether reading them would constitute projects or not is another question.

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[1] "There must not only be a problem, but the individual must see it. The common supposition that problems are recognized is an error. Usually they pass unnoticed." Swift, *op. cit.* (p. 80), p. 10.

[2] "Blind trial and error is the animal and racial way. Unfortunately it continues to be the chief method of modern man. Unreflective adaptation is followed today when obstacles are not so overwhelming as to force deliberation." Swift, *ibid.*, p. 10.

One of the weaknesses of a dull mind is that it fails to see the implications of far-reaching principles and the meaning of fundamental facts. It would be a great achievement indeed if this weakness could be remedied even in part by grouping of subject matter.

There is another type of projects which consists essentially in finding local settings for topics as they are studied or reviewed in the regular courses. The project numbered 8 in the preceding section is a good example of this kind. Such projects are suggested for nearly every subject discussed in this book. See in particular Chapter XI to XXVII. It may be laid down as a general principle that no reasonable opportunity should ever be neglected for making use of concrete, practical material which serves to connect our school subjects with the stream of human life of which we are a part. If such use can be made to embody a problem element, or perhaps more accurately a dramatic element, and thus lay strong hold on attention and interest, so much the better.

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[1] Stevenson, *op. cit.* (p. 93), pp. 233 ff.

Another type of projects, and this is perhaps the one which has been in the foreground of consciousness of many who have written on the subject, is a situation which reveals the need for certain skills, data, or principles not already at hand and which is supposed to create a natural basis for acquiring or mastering them.[1] The project of laying a concrete walk (number 6 in the preceding section) and the attendant study of many matters is an example of this kind of projects. Sir Isaac Newton's attempt to solve certain problems in astronomy and finding the need of mathematical instruments not yet in existence, and his consequent creation of the calculus is perhaps the greatest example on record. This type of projects is probably more liable to misguided use than any of the others. If the project has the dramatic element which creates a real interest in its completion, the ordinary mind will refuse to be long diverted by cognate matters not absolutely necessary. If much preliminary work is required, the whole project is abandoned. Most of us are not Newtons, not even on a small scale. In many cases the project is made the excuse for studying topics simply suggested by it. Such projects remind one of those garrulous talkers whose minds are forever wandering from the point. The project of laying a concrete walk mentioned above is a good example. A healthy, intelligent boy would scarcely regard the laying of a walk as a reason or occasion for studying the topics enumerated on page 98. I have had personal experience laying just such a walk, and I confess freely that it did not even suggest to me the study of those incidental matters. It was necessary to know the proper proportions of the ingredients used and the way to mix them, and this knowledge was promptly acquired. Much of the information needed had been previously obtained by casual observation of such work — observation made with no expectation of ever putting it to practical use. For the rest, we finished the walk. That was our "project"; and when that was completed, we went on with something else, which was not the accumulation of a heterogeneous body of information on concrete.

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[1] Thorndike probably goes further than most educators in denying the efficacy of this procedure: — "Difficulty — temporary failure, inadequacy of existing bonds — on the contrary does nothing whatever in and of itself; and what is done by the annoying lack of success which sometimes accompanies difficulty sometimes hinders thinking and learning." *Op. cit.* (p. 42), p. 277.

It is respectfully suggested that, for one not bound by superstitious fidelity to a formula, the natural way to approach the study of concrete to the extent indicated on page 98 would be something like this: Take note of (*a*) the use of concrete in the immediate neighborhood (there is no need of laying a walk; we have all seen them by the hundred); (*b*) the extent of our country and the use of concrete everywhere; (*c*) the comparative economy of using concrete as inferred from its extensive use in building construction and by railroads; (*d*) the material of which concrete is made and the available amounts of such material; (*e*) the geographical distribution of such material and the cost of distant transportation; (*f*) the probable future use of concrete and its importance in conserving other materials. In short, let the subject of inquiry be announced frankly as "Concrete and its human significance" or something equivalent. The more intelligent the boy or girl the more certain will he or she be to regard the hanging of such a study on the actual construction of a walk as merely solemn pedantry.

Particular attention has been given here to this project of laying a walk because it is believed that it typifies an attempt to make projects carry interest where they do not and indeed should not carry any particular interest. Such use probably has its origin in a desire to make "the modification of conduct the end of all education." I confess frankly that if I understand this statement, I can not subscribe to it. I am under no delusion as to the danger of making this confession. The pronunciamentos on this point are so unreserved and positive that even questioning them may be regarded as foolhardy rather than brave. The following is explicit and to the point:

> Classroom teachers and textbook writers, when faced by the direct question as to the end of education, agree so unanimously with the statement that the end of education should be the modification of conduct, that no notice would need to be taken of any divergent view, were it not for the fact that actual classroom procedure through questions, recitations, reviews, and examinations is frequently dominated by the other ideal, that the mastery of information, and not its applications to problems of conduct, is the important end of education. In fact, this conflict is so real that many of the commonly used concepts enumerated below have been invented for the specific purpose of insuring mastery of information rather than improvement in conduct.[1]

If I am to be educated *solely* for the sake of *modifying conduct*, it must be *my* conduct that is to be modified, for if the contrary view is taken I should be justified in making a minute study of the habits of the African lion *because* the conduct of savages who are obliged to live as its neighbors is affected by such knowledge on their part — a proposition which probably would have no wide appeal. Again the "conduct" in question must be something other than the act of acquiring the information, for otherwise every such act would be its own justification and the proposed formula would be empty words. Finally this "conduct" must in its last analysis be other than intercommunication among human beings about the thing learned, for otherwise the formula would be equally empty; everything that is studied nowadays is the subject of such intercommunication. "Where two or three are gathered together" who are interested in the same subject, there will that subject be discussed no matter how remote from "conduct" it may be. I am quite confident that some other formula than the "affecting of conduct" must be found if it is to cover *all* the ends of education. The simple fact is that we are making great efforts to learn facts which we do not expect to modify the conduct of anyone, except possibly to beget more efforts of the same kind, and it is quite possible that the more intelligent we are the more such effort do we make. How is the "conduct" of an American housewife affected by knowing that cane sugar is produced in Louisiana and not in Illinois? All she needs to know is that, for a certain price, she can buy it at the store, and still this information is one of the objects of a project with unimpeachable credentials. It would be no easy task to show that Newton's Law of Gravitation, or the fact that Polaris is more remote from us than Sirius, has ever modified human "conduct" in the sense in which the word must be used to give a defining meaning to the formula under discussion.

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[1] Stevenson, *op. cit.* (p. 93), p. 11.

It is recognized of course that *one* of the aims of education should be to modify the conduct of the one who is educated; possibly it is the chief aim. But it is contended that there are other aims. Man is possessed of an insatiable curiosity. He is constantly trying to learn more facts about his environment and to explain these facts and for no other purpose than the knowing and the explaining. The fact that such knowing and explaining may lead to more knowing and explaining does not help matters. I once heard a serious proposal that in geometry only those theorems should be studied which are needed in proving other theorems, a proposal which is similar to that famous plan of cutting off the last coach on trains to avoid the danger of rear-end collisions. By following the lead of this curiosity man has incidentally (but not by this means alone) acquired information which has enabled him to greatly modify his environment and thus give us modern comforts instead of the cave of the savage, and to change fundamentally his attitude to his fellow men. But besides this, it has led to a vast body of information which is of appealing interest and which must be regarded as an end in itself. "Man shall not live by bread alone." Much of man's intellectual life must be regarded as an end in itself — one of the final fruits and aims of life. This fact, no doubt, explains the conflict complained of in the passage quoted above.

The writer is far too modest and too wary to attempt a formula which shall include all the aims of education but it is believed that efforts to understand ourselves and our environment is one of these aims. In practice we regard all matters of general human interest as worth while: Peary reached the North Pole; Scott perished in the Antarctic; Stanley crossed "Darkest Africa"; the Earth with its fluid surface is held in a spherical form by the same force that makes us keep company with the Sun instead of shooting off into space. Nor must we assume that such facts as these are of less general or intense interest than other more practical matters. The feat of actually measuring the diameter of Betelgeuze received more publicity than any invention patented during the same year, and the lasting fame of him who measured it is much more secure than that of many Presidents of the United States.

Again each individual is more interested in that which is of general concern. The fact that I build a concrete walk and benefit by its use is of course interesting, at least to me, but a very different and heightened interest is created by the fact that concrete is used for many purposes and in all localities, that the supply of raw material for it is practically inexhaustible and that its use may be widely extended. It may be argued that there is no reason why this should be so, but the simple fact is that it *is* so, and that is the final answer; the fact exists entirely independent of a possible explanation. We shall defeat ourselves if we deny such matters of human interest a place in our education and we shall make a sad blunder if we fail to make use of such interests as primary motivating forces. It is precisely from this point of view that we can condemn most effectively that mummery, which has sometimes passed for education, and which is so well exemplified by the story told by Horace Mann of a class in geography who after a perfect recitation on continents, oceans, islands, etc. were asked whether they had ever seen the earth about which they were reading and unanimously answered that they never had.

48. Origin of the project method. — The project method originated in teaching the elementary phases of such practical subjects as shopwork, agriculture, household arts, and other industrial arts. Its later development has been mainly in these subjects though projects have received much attention in the kindergarten and the lower grades.[1] Stevenson lists 158 references on the project method in which the subject to which the project is applied can be judged from the title. Of these 46 deal with agriculture, 19 with household arts, 41 with shopwork and allied topics, and 52 with all other subjects. The reason is obvious: Instruction in such matters as agriculture especially if the work is quite elementary naturally divides itself into a series of projects. The abstracting of special phases for separate attention is naturally a part of a more advanced and more ambitious plan of study and investigation. These remarks apply to shopwork and the skilled trades of all kinds, and to household arts.

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[1] One of the most complete collections of projects for any particular school situation is to be found in: Krackowizer, *Projects in the Primary Grades;* J. B. Lippincott Company, 1919. This book contains also a valuable and interesting discussion of the project method for these grades.

The type of situations in which the project method was first developed serves to explain its early character, as its later spread to other subjects accounts for the change which the idea as to what the project method really is has undergone in the last few years. At first there was much emphasis on manual performance. It was assumed, more or less explicitly, that a project was a practical piece of work involving considerable physical labor and, more or less incidentally, some thought. At this stage, the care, feeding, and weeding out of a flock of chickens would surely be regarded as a project, but it is not so certain that keeping a set of cost accounts for a farm would be a project. Later, as some teachers began to organize projects in such subjects as English, arithmetic, geography, and history, the idea of what a project really is came to be modified. There was less insistence on manual performance. The project might be a purely intellectual matter with such minor manual adjuncts as writing or drawing.

When one notes the type of subjects in which in the main the project method has been developed, one very naturally questions its value in the teaching of other and very different subjects. It may be the very best method possible for teaching shopwork or agriculture and still be a failure in teaching history. The only way to find out is to give the method a thoroughgoing trial in these other subjects; and this brings us to the subject of the next section.

49. Judging the success of a new method. — It is easy to be deceived as to the real value of a new method. It happens very seldom that two methods are tried out judiciously by the same teacher or by different teachers under similar conditions. The new is usually tried by those who believe in it before they try it, and for this reason a final report of success is practically assured. The enthusiasm with which the teacher works in such a situation would insure a high degree of success no matter what method she used. Furthermore, her own estimate is probably colored by her expectancy. If, on the other hand, a teacher is induced to try a plan about the success of which she is skeptical, failure is usually insured from the start.[1] Hence the vogue and reputed success of any new method which is not altogether preposterous depends in no small degree upon the enthusiasm, persistence, and personally engaging qualities of its sponsors. The earnest missionary zeal of him who offers a "new plan of salvation" has been responsible for quite extended use of many methods now relegated to their proper place but which once were "the very latest" and the only thing "up-to-date." The Grube Method, the Spiral Method, the Speer Method, the Montessori Method, and a host of other methods "have had their day and (relatively speaking) ceased to be." Each in its day was urged by its adherents as the *sine qua non* of educational progress. Each was advocated most vociferously and persistently until another *sine qua non* was found, when these enthusiastic souls flocked to the new banner. The writer ventures the prophecy that we are now on the eve of a period in which "habits" and "bonds" will be the catchwords. By those now mainly responsible for these words they are used conservatively and sanely but, unless historical precedents deceive us, we may expect a period in which the educational propagandists will urge as the only possible achievement of our schools the formation of "habits" and "bonds" in a narrow sense, and that all efforts at generalization are futile. While the great majority are conservative, are glad to let well-enough alone, and to stagnate without any disturbing thoughts of improvement or change, there is a minority who always regard that which is as wrong, and who see in the distance that which at last will give us a "new heaven and a new earth." This is not at all peculiar to the profession of teaching. Democracy, extension of the suffrage, initiative and referendum, recall of elected officers, the cults and the isms, different patent medicines, no medicine, vitamines, fresh air, these and many others have at one time or another so filled the mental vision of many people as to make all else seem trivial and of minor consequence. There is no gainsaying the service which these dynamic spirits render to the cause of progress; the only difficulty is that they often lead us in the wrong direction, and that under taunt of being "behind the times" we not infrequently follow even when reflection would convince us that we are going in a direction from which presently we shall all have to return. The confidence and enthusiasm of the leaders are such as to preclude thought and reflection except on the part of those who by nature are skeptical and not easily led. These are considerations which must be borne in mind when judging new methods and points of view. The history of the teaching of arithmetic furnishes ample warning.[1]

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[1] "If a man expects a plan to succeed the chances are that he will carry it through, and if one anticipates failure one is quite certain to be gratified." Swift, *op. cit.* (p. 80), p. 14.

There have been only a few fundamental discoveries and inventions. Some twenty-five years ago the writer saw an automobile for the first time. It was a crude affair rattling along some ten or fifteen miles an hour and breaking down on an average every twenty-five miles. The difference between that car and the present motor car was brought about, not by any startling invention or discovery, but by a long succession of minor adjustments and improvements, no one of which would be of notable significance to most of us. It may be that the real improvements in teaching come about in much the same way. One danger is that, in our effort to follow the leaders, we may neglect that quiet painstaking work of detailed improvement and adjustment which is essential in all solid progress. Not the new because it is new nor the old because it is old, but a dispassionate judgment of both and a wholesome regard for the value of that which has been tried and whose shortcomings have been largely eliminated, together with a clear conception of the work that is required to fit the new into our workaday world — these are the qualities which would seem to be needed to judge properly the value of a new method.

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[1] Thorndike, *op. cit.* (p. 42), pp. 3–7, describes the overemphasis which within the short space of forty years has been put on the "series or counting idea," the "ratio idea," and the "relational idea."

David Eugene Smith, in *The Teaching of Elementary Mathematics;* The Macmillan Company, 1900, pp. 71–97, describes cyclic changes in the methods of teaching arithmetic.

50. The child's "natural setting." — It is asserted that the "natural setting" of a problem is the setting in which it occurs in life outside the school. On this point we must be clear. The "setting," it is said, must be that of the world in which the problem arises as a necessary part of carrying on its work. Confining ourselves to arithmetic we may say with confidence that the natural settings of problems as they occur outside the school are the affairs of adults — home making, industry, business, and the professions. To make our point a digression is necessary.

Man has two inheritances: He is born into the world with a certain bodily structure and with certain capacities for reaction to stimuli and for growth. These are his biological inheritance which depends upon the character of the germ cells from which he sprang and no doubt in some part upon his prenatal nurture. The type of physical and social environment in which the accident of birth places him determines in a very large measure the habits he will form, the thoughts he will think, and the relative valuations he will place on the things of life. It is difficult to exaggerate the possible differences: Shall he grow up in a world unmodified by the arts of civilization; his daily food the berries, fruits and herbs of uncultivated fields and forests or the game and fish which he kills with his unaided hands or with the crudest of weapons; his shelter a cave in the rocks or at most a tent; the forces and appearances of nature a constant source of superstitious dread, where "The hungry stars in heaven like eyes of wolves glared at them"; famine always near; an enemy always in ambush; his children crushed before his eyes by ravenous jaws; shivering as the snow falls on his half-naked body; and as disease or old age comes, die of neglect and starvation unless a hungry beast comes to end his miserable existence? Or shall he live in well-appointed homes in London, Paris, or New York; wear comfortable and protecting clothing; eat dainty fare that is always ready at the appointed hour; attend the opera and the theatre; travel on steamships and railways, or in automobiles and airplanes; read Homer, Shakespeare, Robert Burns, Kipling, or Mark Twain; have delightful companionship with his fellows; look on nature unafraid — her undisputed master — the beasts of the field and the birds of the air his servants or his toys? Shall he learn the few tricks of the savage and spend his life in alternating between drowsy loafing and the strenuous chase and the life and death fights with his fellows; or shall he acquire the multitudinous habits and skills necessary for civilized life; become adjusted to well-ordered, unexciting, industrial toil and learn the innumerable things and acquire the tastes that constitute the equipment of the cultured man of civilization? The answer to these questions depends upon the accident of the environment into which he is born. This is his racial inheritance.

There is little evidence that the biological inheritance of man has changed greatly since he was a savage. There are plateaus in the course of evolution where for a long time there seems to be little change. During historic times it seems that man as an individual has been on such a plateau. The savage and the civilized man are not much unlike at birth. At twenty-five they are tremendously different. Now, that which distinguishes civilization from savagery is comparatively very recent in this world, and there is no reason to believe that biological inheritance has as yet provided special adaptation to the new conditions; though we may well believe that such adaptation had been made to savage life which was of infinitely greater duration, and in which the law of "survival of the fittest" (those that were best adapted) worked with much greater effectiveness through the ruthless destruction of the unfit. No doubt the education for savage life is more "natural" to the child than the education for civilization. He has the biological inheritance of the savage. It is proverbial that the young child *is* a savage. But environment molds him gradually but inexorably so that at the end of two decades he is a civilized man or woman. When the little savage comes to the school at the age of six, modern industrial situations are far from being "natural" to him. We shall never deceive ourselves more grievously than by assuming that the "natural setting" in modern life is necessarily a fortunate setting in which to teach anything to the young child. On the other hand we must not forget that this savage is in the *process of becoming civilized*, and that as the years go by his original savage nature is gradually molded into the form required by his environment. But even this does not go on as rapidly as one might think. The real adaptation to the conditions of a life of industry comes only after the individual has been immersed in it in the actual work of adult life; and the full appreciation of the word "practical" comes only then. If it is urged that problems must be given to young children in their "natural setting" for the purpose of developing skill in performing a task in precisely the manner in which we believe the child will be called upon later to perform it, there is no room for argument; but if this "natural setting" is urged because it will make the problem interesting to the child we must give our assent with no small misgivings and with many reservations.

One more point must be made. To a large extent school life is *the* life of the child; and make school as bad as we will, it will not be more "unnatural" to him than the ordered life of monotonous work which is the price that adult man pays for the comforts of civilization. It is in the school that the child has his tasks, his trials of strength, and his rivalries. To him the tasks of the schoolroom become ends in themselves or means to the rewards which he desires. Drill in long division is just as "natural" and not one whit less distasteful than setting tables, sweeping floors, husking corn, pulling weeds, or any other work if continued beyond a short initial period which may be called the "play period." The school group, doing the same kind of work, measured by the same standards, and reaping the same rewards which are supposed to be meted out according to the character of the work done, is a real social unit and its life is less artificial to the child than is that of a group of adults. To achieve the best possible result in the schools it would seem best to lay aside many of our simple and therefore necessarily inadequate formulas and make a naïve, direct study of the child as he is in each stage of development from the savage at six to the semicivilized at fourteen or sixteen. If we can find his real interests, the stimuli to which he responds with all his might, and make these the vehicles on which to carry to him the educational work which we want him to do, we shall make real progress. This topic is continued in Chapters III and IV of this book.

51. Extent to which projects can be used in arithmetic. — The reader who follows this book through to the end will find that a multitude of projects are proposed without calling them by this name. If we are to regard as real projects all those given on pages 98–100 (with the exception of the first two they are vouched for as "true projects" by earnest advocates of the project method) and do not take too seriously the characterization given on page 94 we shall have little trouble in putting nearly all of arithmetic into a dramatic setting which will make it a series of projects, without interfering in the slightest with the topical arrangement of the subject matter. If answering the question, "What part has transportation played in developing our country?" is a true project, it is equally a project to study practical uses of discounts, of commission, of plain percentage, or of any of the other topics to which arithmetic is applied provided each of these studies is undertaken with a purpose which the student has made his own. Under skilful use of the motivating elements described in Chapters III and IV, many projects will be undertaken by the children themselves. When a boy has decided that to know the multiplication table is highly important for him and resolves to master it at all costs, this becomes a true project (compare project numbered 4 on page 98). Similarly the learning of all the operations and the drill on them may be made into real projects.

To repeat from a former section, the word "project" was first proposed in connection with certain units of work in such directly practical subjects as agriculture. In these projects manual work was an essential element. When the word came to be applied to undertakings in such subjects as history, geography, arithmetic, and reading, the manual-work element came to be regarded as unessential. Making a complete "tax spread" is as much of a project as making a sled or handling a flock of chickens. The project must involve something more than the solution of a single stated problem as understood in arithmetic. Making an effort to remember $7 \times 8 = 56$ would not be regarded as a project while the effort to master the multiplication table would be so regarded provided this effort is properly motivated. (See projects numbered 3 and 4 on page 98.) If we now regard proper motivation from the point of view developed later in this book, we shall have no trouble in devising projects for practically every topic in arithmetic. The weakest element in the project, as applied to grade work, has been the more or less explicit insistence that the motivating force must be the practical usefulness in the industrial world of that which is learned.

Finally, we must avoid projects which carry us too far afield into extraneous matters. It has been proposed that the child should learn the meaning of ½, ⅓, and ¼ by doing complicated pieces of shopwork (for the child). For one thing the child knows the meaning of these fractions much earlier than is frequently supposed and for another the shopwork would so tax his energy and attention as to obscure the fractions which it is desired to study. We must also avoid making a comparatively simple operation the excuse for endless excursions, such as is exemplified by the concrete-walk project on page 98. If we can arouse the child's interest in the accomplishment of a purpose, we shall have the material for a project. As the child grows older, that which is of general human significance is of increasing interest to him. Such interest should be encouraged and free use made of it in laying the basis for projects in school work.

REVIEW QUESTIONS

1. What general reasons have we to believe that conditions in the schools are more conducive to intellectual activity than are the conditions outside the schools? (§ 34.)

2. Describe some of the ways in which children learn outside the schools. Are situations frequent in which a problem arises which causes the child to be conscious of his shortcomings and which thus causes him to strive to overcome such shortcomings? What are the motives which cause young children to learn outside the schools? (§§ 35, 36, 37.)

3. Discuss the learning by adults (a) as to the things learned; (b) as to method of learning. (§ 38.)

4. Discuss the statement that the learning by the average adult is empirical. (§ 39.)

5. In what respects should life in the school be made to differ from life outside the school? (§ 40.)

6. Discuss the ways in which the scientist learns new truths. Why is the method of the scientist different from that of people in business for example? (§ 41.)

7. Describe the topical method of studying. Is the topical method of studying used generally outside the schools? (§ 42.)

8. State at least three adverse criticisms of the topical method. (§ 43.)

9. What answers can be made to the criticism just stated? (§ 44.)

10. Give the definition of project quoted from Stevenson on page 94, and state more fully each of the conditions specified in this definition. (§ 45.)

11. Do the projects described on pages 98–100 possess all the elements included in Stevenson's definition?

12. Describe the type of projects represented by No. 1 on page 98. (§ 47.)

13. Describe the type of projects represented by the question: "What part has transportation played in developing our country?" (§ 47.)

14. Describe the type of project represented by No. 5 on page 98. Give a criticism of this type of projects. (§ 47.)

15. Discuss the statement that "the end of education should be the modification of conduct". (§ 47.)

16. In the teaching of what subjects did the project method originate?

17. Describe the difficulties of deciding what method of instruction is the best. (§ 49.)

18. Discuss the general tendencies (a) of the overconservative; (b) of the restless or radical. (§ 49.)

19. What is meant by the "natural setting" of a problem as this expression is used in current writings? (§ 50.)

20. Discuss the statement: "The setting of a problem in practical life may be a highly unnatural setting for the young child". (§ 50.)

21. Discuss the school as the community center for the child. (§ 50.)

22. Discuss the extent to which projects may be made the basis of instruction in arithmetic. Would this necessitate changing the order in which the topics of arithmetic must be considered? (§ 51.)

CHAPTER III

Motivation in the Early Grades

"One point must still be greatly dark,
The moving why they do it."

52. The usefulness of arithmetic not a motivating force. — We have given arithmetic a place in the curricula of our schools, not because it is necessary for the living of a full and normal child life, but because it is of practical use to the adult. Whatever we may say about school being a part of life, the simple truth is that, for practical purposes, we regard it to a large extent as a preparation for later life. The content of the curriculum is largely determined not by the present needs of the child but by his supposed future needs. We take a view of the whole of life and then attempt to teach children facts and processes and to develop in them habits which, in our opinion, will be useful somewhere on the journey from the cradle to the grave.

To the child this preparation for a life which to him seems far off is of little or no importance.[1] People of mature years find it difficult enough to forego the satisfaction of present wants for the sake of greater future welfare. How much more is this true of the child! The boy of ten is farther from twenty than the young man of twenty is from sixty. Not infrequently attempts to motivate the study of arithmetic in the early grades have been based on the assumption that children *are* in fact what we ourselves think we *ought to be.* Indeed, it must be regarded as fundamental that the young child has no keen interest in anything just because it is useful to the adult. He is not concerned about preparing for success in life. The prospect of making a living for wife and family is not a motivating force to the boy of ten; the fact that numbers are needed in the work of the housekeeper does not make the girl in the fourth grade interested in arithmetic.[1] The conclusion is:

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[1] "The pupil of the lower elementary school is more or less a creature of the present. His dominant interests are in things that appeal to him because of immediate utility or pleasure." Brown and Coffman, *op. cit.* (p. 31), p. 349.

The reasons which have caused us to give arithmetic a place in the curriculum are not reasons why the child will want to study it.

53. The nature of interest. — The word "interest" is used in at least three different senses: Certain experiences satisfy us, seduce us to acquiescence, or engross our attention — simply because we ourselves are what we are. Such experiences are said to be *interesting* and the condition of mind while attending to them is said to be one of *interest.* An agreeable experience causes a desire to reproduce it, and hence we are anxious to return to the situation in connection with which it came. This anxiety to return to a situation is a relatively permanent condition of the mind which is called *interest in that situation.*[2]

The word "interest" is also used to describe a state of mind in which a person wants to perform certain acts, not because they are attractive or agreeable in themselves — they may be distinctly disagreeable — but because they are believed necessary in order to secure ultimate ends which are desired. The work of checking the references in this book has no attraction for its author, but is necessary to insure their accuracy. So he tells the librarian that to-day he is "interested" in such and such volumes.

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[1] The doctrine urged so vigorously in some quarters about a generation ago that only those subjects should be taught which are of natural interest to the normal child now seems to be given up quite generally. The following excerpt from Thorndike, citing Stanley Hall, is significant: "Finally it must be said that under the pressure of obvious facts even the most ardent advocates of Nature's infallibility always somewhere give the doctrine up. So Stanley Hall writes:

> . . . 'now another remove from nature seems to be made necessary by the manifold knowledges and skills of our highly complex civilization . . . the child must be subjected to special disciplines and be apprenticed to the higher qualities of adulthood, for he is not only a product of nature, but a candidate for a highly developed humanity. To many, if not most, of the influences here there can be at first but little inner response. . . . The wisest requirements seem to the child more or less alien, arbitrary, heteronomous, artificial, falsetto.' (Preface, p. xii.)" Thorndike, *op. cit.* (p. 21), Vol. I, p. 279.

[2] "Spontaneous attention may be a primary mental activity. Interest is always secondary. It is a conscious phenomenon attaching to objects of which we have already had some experience." James Rowland Angell, *Psychology;* Henry Holt and Company, 1904, p. 421.

For the present we are primarily concerned with the first two of these types of interests. In these senses of the word, interest must be conceived as a primary fact based directly upon our mental constitution. We like to do one thing and dislike to do another, and perhaps we feel that we can not give reasons for our likes.[1] There are certain stimuli to which we respond directly and without hesitation. The child plays and the glee of activity beams on his face. There is never a question but that to him play is supremely worth while. He listens to stories or looks at a strange animal with a spontaneity that arises from the depths of his nature. These are examples of what may be called our primary or immediate interests.[2]

54. Motivated acts: interest derived from the immediate present. —Nearly all our purposed acts are highly complex, and very few of them are a "pure delight." The attractive and unattractive experiences of life are so inextricably mixed that we are constantly performing acts which in and for themselves we would rather leave undone. In some cases the motive for doing a disagreeable or unattractive act arises out of the immediate present while in other cases it arises from remote considerations, such as expectation of future personal reward or desire to leave a competence to one's family. We shall first consider those that arise out of the immediate present: The nature of such interest may readily be understood from examples. Recently I saw a small boy ride a tricycle over a concrete walk, on which little stones had been placed. The boy dismounted, removed the stones, and resumed his riding. He was primarily interested in riding the tricycle; but, in order to be able to achieve the object of that interest, he found himself obliged to do another act which in and of itself did not at the time have the slightest interest for him. If the boy had been asked to take the stones off the walk without any reference to the riding of the tricycle, he would probably have found the task irksome. However, when made the condition of riding the tricycle, in which he was greatly interested, the removing of the stones was done willingly and without any sense of hardship.

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[1] "Those things to which we spontaneously attend are the things to which our minds, by virtue of their temporary condition, inevitably go out." Angell, *op. cit.* (p. 120), p. 89.

[2] "There are cases where self-expression is direct and immediate. It puts itself forth with no thought of anything beyond. The present activity is the only ultimate consciousness. It satisfies in and of itself. The end *is* the present activity, and so there is no gap in space nor time between means and end." Dewey, quoted by W. W. Charters, *Methods of Teaching;* Row, Peterson and Company, 1912, p. 151.

"I know that the nest full of eggs is an 'utterly fascinating' object to the broody hen, that scarlet is hatefully interesting to bulls and entirely indifferent to sheep, that I cannot keep my eyes from the waves that are breaking on the shore, whereas my neighbors are tranquilly playing cards or making Battenberg lace. Biologically, I may assign a reason for each of these interests, but psychologically, from my own introspection or inference, they are inexplicable. The hen attends to the eggs, the bull to the scarlet parasol, I to the beating waves, — because we must; we are interested — because we are interested." Mary Whiton Calkins, *First Book in Psychology;* The Macmillan Company, 1906, pp. 139 f.

Many good people have a keen desire to take a trip to Europe though they know that there are many elements connected with the journey which are not of the slightest interest and indeed are quite the opposite of enjoyment. There may be a long trip on a hot, dusty train and possibly an ocean voyage made exceedingly disagreeable because of seasickness. There are likely to be moments of embarrassment and even positive misery when one is traveling in a strange land. These disagreeable features, however, are so closely bound up with those in which there is real interest that they are not thought of as things apart. As a whole, it is decided that the trip is delightful and that whatever untoward and unenjoyable things may be connected with it are to be taken as matters of course.[1]

55. Motivated acts: interest derived from remote considerations. — The acts of a mature person are frequently motivated by considerations which lie far from the acts themselves. He decides that for certain reasons this or that is the thing to do, and then he goes to work and does it, following up one kind of work for weeks, months, and perhaps years and never permits himself to consider seriously whether or not he likes to do it. There are certain things which we desire but to get these things we find that we have to pay a price, possibly a price which we do not at all like to pay. We have to toil for our food and our clothing and our shelter. A farmer is not interested in plowing and seeding and harvesting in and for themselves but he has a keen and vital interest in the grain or in the things which the grain will bring him; and the moment that the plowing and the harvesting are made conditions precedent to the enjoyment of these things, he acquires a motive for doing them. Frequently the really effective workman is surprised when asked by a tyro whether or not he likes his work. He does not admit this as a relevant question. He knows that the job should be done, that it is important that it be done, and he has decided to go ahead with it and is doing it with might and main. The question as to whether or not he likes it is never permitted to come up for serious consideration.

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[1] "Interest is immediate when the topic or stimulus is interesting in itself, without relation to anything else; derived, when it owes its interest to association with some other immediately interesting thing. What I call derived attention has been named 'apperceptive' attention." James, *op. cit.* (p. 57), p. 221.

"On the other hand, we have cases of indirect, transferred, or technically, mediated interest. That is, things indifferent, or even replusive in themselves, often become of interest because of their assuming relationships and connections of which we are previously unaware." Dewey, quoted by W. W. Charters in *Methods of Teaching;* Row, Peterson and Company, 1912, p. 151.

"There are many facts in each branch of study which, in themselves, excite little or no interest, just as there are many details in a man's business which, in themselves, are only tedious. All of these facts may acquire a secondary interest by close association with interesting things with which they are brought into relation." Charles A. McMurry, *The Elements of General Method;* The Macmillan Company, 1903, p. 91.

The child's acts are not frequently motivated by remote considerations. He does not decide that this or that is to be done and then go ahead and do it persistently for a long time. The thing which actuates him lies nearer home. It may be that authority pushes him into doing things, that he is caused to do them by imitation, that immediate rewards entice him; but a far-reaching purpose relentlessly followed up is never his. One may get a child to save money by certain immediate considerations, but the idea that he is to have the use of it throughout his whole life, that it is to bring him a constant and increasing income will not make him persist in saving. A small child may *say* that it is worth while to learn to read because he hears it said that a person who does not know how to read is not "much good," but that does not go very deep. It develops in him no permanent enthusiasm for learning to read. His statement is perhaps not much more than a simple repetition of words which he has heard others use and which he thinks they want him to make. We too frequently deceive ourselves into believing that children really feel what they say when they repeat the formulas they have heard from their elders.

There is one fundamental difference between motivation derived from the immediate present and motivation derived from remote considerations which should be noted. When the motive lies in the immediate present, there is less likely to be a struggle and a feeling of privation and hardship. Distasteful exertion making possible present enjoyment or the escape from present danger does not occasion the struggle and feeling of misery that are experienced when the same exertion is caused by remote considerations. It always requires effort, often very disagreeable effort, not to discount the future too heavily. This is a type of effort for which the child has little or no capacity, a fact which is too frequently neglected in educational theory.[1]

56. Definition of motivation in elementary grades. — We shall now attempt to define motivation of arithmetic in the elementary grades bearing in mind that most of the work which must be done in a graded-school course in arithmetic is such as would not be done by the child on his own initiative, that is, from his own spontaneous or primary interest.

Motivation in elementary work consists in so combining those activities which are desired of the child with other activities in which he is spontaneously and directly interested that the combination of activities will to him be interesting and attractive.[2]

This definition is elastic. It includes the use, not only of the child's early, spontaneous interest, but of any interests which he may have acquired at the time a certain act is to be motivated.

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[1] "Evidently this [to sacrifice the satisfaction of a present want for the sake of future welfare] can only occur when we have developed intellectually to a sufficient degree to set over against some momentary disposition, or action, a more or less definitely formed plan involving interests and purposes opposed to the present activities. . . .

"The act of voluntary attention is, in short, an expression of the sovereignty of the whole mind over its lesser part, *i.e.* over the disturbing or alluring ideas and sensations." Angell, *op. cit.* (p. 120), p. 89.

[2] "It is only by a certain amount of artifice that we can enlist other vehement inborn interests of childhood in the service of arithmetical knowledge and skill." Thorndike, *op. cit.* (p. 42), p. 195.

"Educational Corollaries. First to *strengthen attention in children* who care nothing for the subject they are studying and let their wits go wool-gathering, the interest here must be 'derived' from something that the teacher associates with the task, a reward or a punishment if nothing less internal comes to mind. If a topic awakens no spontaneous attention it must borrow an interest from elsewhere. But the best interest is internal, and we must always try, in teaching a class, to knit our novelties by rational links onto things of which they already have preperceptions. The old and familiar is readily attended to by the mind and helps to hold in turn the new, forming, in Herbartian phraseology, an *apperceptionsmasse* for it. Of course the teacher's talent is best shown by what *apperceptionsmasse* to use. Psychology can only lay down the general rule." James, *op. cit.* (p. 57), p. 236.

"The third element in rational use of nature's capital of motives is to exercise ingenuity in attaching and detaching satisfaction and discomfort to and from this and that particular situation or feature of a situation. The genius at human engineering will learn to apply those forces with a skill like that whereby the mechanical and chemical engineers use the forces of gravity and atomic affinities." Thorndike, *op. cit.* (p. 21), Vol. I, p. 296.

In order to organize motivation as suggested by this definition, it is necessary to study in detail the child's spontaneous interests. This we now proceed to do:

57. Play — muscular activity. — One of the earliest and perhaps the most common of the child's spontaneous interests is that of mere play, meaning by play mainly muscular activity, such as running, yelling, etc. But this muscular activity is not absolutely at random. It tends to take the form of games in which other elements of the child's nature begin to show themselves, make-believe being ever ready to make the situation real.[1] This instinct for play may be made to contribute in making school work attractive, especially in the lower grades. Make-believe may even take the place of actual physical acts. A child climbs a number ladder much better if urged to climb "as fast as ever he can" and if making a mistake means "falling off the ladder and hurting himself." Numbers arranged around a circle are added more rapidly and more accurately if it means "running" around the circle and if making a mistake means "falling." A boy in naming the sums of number combinations on a page in the number primer increased his speed by playing that they were stones in a muddy place and that he was stepping from one to another. When he made a mistake, he fell and "got mud all over." [2]

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[1] "Thus, playing with a doll is partly imitation of the mother, partly an early budding of the maternal instinct; the hunting and fighting plays, the constructive plays, all forms of rivalry and competition are but the exhibition of different instincts under make-believe conditions, under circumstances assumed to exist for the sake of play." W. B. Pillsbury, *The Fundamentals of Psychology;* The Macmillan Company, 1916, p. 438.

[2] "The value of play in developing capacities through practice in advance of the actual necessity is obvious." Pillsbury, *ibid.*, p. 439.

"There are three sorts of motive for drill. One motive is interest in drill for the fun one can get out of it. This is probably stronger than most writers on methods of teaching will admit and not so strong as most teachers assume. Children do enjoy retraversing something, if it happens to be interesting. They sing ditties over and over again, they ask for the same story and admit no variations." Charters, *op. cit.* (p. 123), p. 387.

58. The child's desire to construct things. — Most children have a desire to construct objects resembling those they have already seen. If a set of blocks is given to the average boy or girl, the child will begin to arrange them to represent houses, ships, automobiles, etc. (see section 65). This interest in construction may be used to motivate number work. "How many sticks are needed to form a triangle?" "To form two triangles?" "Who can form two triangles with five sticks?" "How many sticks are needed to form three triangles?" "Who can form three triangles with eight sticks?" "With seven?" Such questions may be continued, culminating in: "How many sticks are needed to form six separate triangles?" "Who can form six triangles with seventeen sticks?" "With sixteen?" "With fifteen?" "With fourteen?" "With twelve?" (Six triangles can not be formed with thirteen sticks of equal length, if they all lie in the same plane.) Similar questions may be asked on squares and rectangles. Again care must be taken that the play element does not submerge the number element and that a keen desire for correct answers is maintained. However, the solving of the puzzles is of real value, since it has many elements in common with the solution of all problems — prolonged attention, enterprise in seeking out possibilities, and industrious persistence (see section 9).[1]

59. Personal distinction and rivalry. — One of the keenest of the child's interests (and of the adult's as well) is that which arises from personal rivalry. If John can show that he is better than Harry in some respect — is stronger, can run faster, or can outdo him in any way which he thinks worth while or which he knows others think worth while — he will derive therefrom the keenest satisfaction and enjoyment.[1]

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[1] "Several attempts have been made to so build the course of study that [at least in the early grades] education shall come in part through building things. For, in order to build, it is necessary to learn arithmetic, to draw, to talk, and to think. Particularly is thinking better, because they think in terms of concrete cases which they understand, rather than memorize abstract matter that is foreign to them." Charters, *op. cit.* (p. 123), p. 130.

60. The fighting instinct. — The spirit of conquest helps not only in overcoming personal enemies, but also in conquering material, intellectual, and moral obstacles. "I'll show you" can be addressed as effectively to a stubborn task as to a living individual.[2] In some children it is easy to arouse the "I'll show you" attitude towards a trying task, an attitude which may lead to a very deep-rooted determination to succeed.

61. The group or community instinct. — Children, as well as older people, like to run in flocks. That which the group does, each one has a well-nigh irresistible tendency to do. In these latter days we have heard much about mob psychology and about the need of strong characters which shall be able to stand out and not follow the mob. We are now concerned simply with the fact that the tendency of the individual to do as the group does is one of the fundamental inclinations of life and that it is at least as strong in the young as it is in the old.[3]

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[1] "Except for him [Darwin] little attention has been paid to the originality of the hunger of man for the externals of admiration and the intolerability of objective scorn and derision. Yet these forces of approval and disapproval in appropriate form from those above and those below us in mastery-status are and have been potent social controls. For example, the 'discipline' of a humane home or school to-day relies almost entirely upon such approval from above, and finds it even more effective than severe sensuous pains and deprivations." Thorndike, *op. cit.* (p. 21), Vol. I, p. 89.

"Social discontent comes from the knowledge or fancy that one is below others in welfare. The effort of children in school, of men in labor, and of women in the home is, except as guided by the wise instinct of nature or more rarely by the wisdom of abstract thought, to rise above some one who seems higher." Thorndike, *ibid.*, Vol. III, p. 312.

[2] "The fighting instinct is in fact the cause of a very large amount of the world's intellectual endeavor. The financier does not think merely for money nor the scientist for truth nor the theologian to save souls. Their intellectual efforts are aimed in great measure to outdo the other man, to subdue nature, to conquer assent." Thorndike, *ibid.*, Vol. III, p. 203.

[3] "To-day we hear not a little of 'the laboratory method' and 'individual teaching' — a return to the methods of the past, methods in which the inspiration of community work was wanting, methods long since weighed in the balance and found wanting." D. E. Smith, *op. cit.* (p. 111), p. 75.

62. Group rivalry. — Groups develop rivalry among themselves which is probably keener and more intense than the rivalry among individuals. One phase of group rivalry is that between nations, and we have seen people by the millions go unhesitatingly to their deaths under pressure of its compelling stimulus. Players on football teams work harder because of group rivalry than they would ever work for pay or for personal advantage. This type of rivalry is easily stimulated among groups of children and may be made one of the most potent of the forces which cause children to conquer especially the early parts of their school subjects. Group rivalry is considered further in sections 75–78.

63. Interest in things already known. — It is one of the fundamental laws of psychology that persons are interested in what they already know or at least know in part. It is stated that the natives who first saw the ships of Columbus were highly interested in the small boats which he used but that they took the ships themselves for granted. The small boats made contact with the things with which they were already familiar, while the ships were so entirely unusual as to almost escape attention.

Recently the writer found he could put himself to sleep by attempting to read an abstruse book on the philosophy of mathematics. One night he ran into a series of chapters dealing with topics in which he had earlier taken keen interest and with which he was familiar. There was no going to sleep that night. Reading page after page and chapter after chapter, as the old

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"Another still later tendency is to have the pupils coöperate in groups. This is in opposition to the individualistic attitude of the traditional school which requires pupils to work by themselves and to be held responsible for a lesson only to the teacher." Charters, *op. cit.* (p. 123), p. 131.

"Finally, and most important of all, the social instinct supplies the desire to be popular, to seek the approval of the fellows upon which depends the force of social convention, and which drives to work when individual need and individual instinct exhaust their impelling power." Pillsbury, *op. cit.* (p. 126), p. 442.

and familiar, treated in a fresh manner, now with trenchant criticism and now with interesting emendations and extensions, riveted the attention and aroused the mind to activity, he was led to do several hours of aggressive and agreeable work instead of gradually dozing off into a night of peaceful rest. Now he is more careful about the books he reads after retiring. The reason that some chapters in this book produced sleep while others produced wide-awake activity (there may be differences of opinion as to which was the more valuable) lay entirely in the fact that the reader happened to be familiar with the subject matter of the latter and not with that of the former.

64. Interest in what is new. — It is a well-known fact that in order to sustain interest certain new elements must be introduced. Thus continuous drill on number combinations all of which are known will grow tiresome more quickly than it will if intermixed with the learning of new facts about number combinations. The chapters which failed to put the writer to sleep failed to do so not only because they treated familiar matter but because they introduced elements of novelty. The judicious mingling of the old and the new is one of the chief arts in instruction. In case prolonged drill is necessary on subject matter all of which is familiar, the new elements may be introduced by varying the form of the drills and the motives used.[1]

65. Imitation as a basis for interest. — In a very important and vital sense, human beings are social animals who imitate the actions of those about them. There is a direct tendency on the part of the child to do that which he sees others do. Of course, the acts to be imitated must be reasonably within his power of imitation. In this manner, environment controls the child's interests to a large extent.

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[1] "Furthermore, in the light of our preceding analysis, it seems clear that the interest which we are said to feel in strange things finds its basis in the expansion of ourselves. Not the absolutely strange thing do we find interesting, but the thing familiar enough to be vitally connected with our past experience and still novel enough to be felt as a definite enlargement of this experience." Angell, *op. cit.* (p. 120), p. 426.

The reason why the child imitates is obvious. His whole nervous and muscular anatomy is such as to make him cry out for activity. He likes to be doing things all the while. There is, moreover, a desire for a certain complexity in what is done. It is not enough that he merely exercises his muscles; he wants to exercise them in some purposeful and orderly way. The mind of the child demands that it be placed at the helm and that it be given exercise in directing the activities of the body according to a more or less clearly preconceived plan. The only way, however, that the child knows to do this is by imitating what he has seen. The child must play as he has seen others play, he must build what he has seen built. Give a young child a set of blocks, and he will build houses if he has seen houses; he will build ships if he has seen ships or pictures of ships. Imitation is fundamental in the transmission of the acquirements of one generation to the next. We learn the manners, the speech, the ways of doing a thousand and one things from the generation that precedes us. Imitation, however, seems to direct rather than to create interest.

66. The pleasure of activity. — Any activity which is adapted to the child's power and skill tends to create a pleasurable sensation, and any activity which develops a pleasurable sensation, unmixed with deterrents, creates interest. This type of interest is what people call "love of one's work." It is one of the most prominent and salutary of interests and is one of those which we should seek ultimately to develop, no matter what other more evanescent interests may be required in the earlier stages, or what more general purposes may be the mainsprings of our actions. The ideal relation of the individual to his work is that the doing of it should be a source of pleasure to him. Pleasure in one's activity is conditioned mainly on success in doing that which is difficult enough to present a challenge. To demand of a child the solution of one hard problem may result in an entirely unsuccessful attempt and then in a feeling of aversion to the whole subject. A series of well-graded problems, simple enough that the child can solve them and still difficult enough to test his skill and mettle is likely to result in a feeling of satisfaction, keen interest, and in eagerness for more work of the same kind.[1]

The following excerpt from Thorndike states the facts admirably and at the same time suggests a psychological explanation.

> The hypothesis that man's brain contains many neurones in "readiness to act" besides those whose action is concerned in the behaviour-series of the specific instincts must, I think, be carried further. There are not only neurones ready to be set in action by direct stimuli from the sense-organs, but also neurones ready to be set in action by more remote or secondary connections. For example, a baby likes not only to see a pile of blocks tumble or a wheel go around, but also to find the blocks tumbling when he hits them, or the wheel revolving when he pushes a spring. Satisfactions of the second sort are, indeed, if anything the more potent. Merely hearing the toot of a horn is a feeble joy compared to blowing it. Now "tumbling when I hit them," "whirling when I push" and "tooting when I blow" are samples of secondary connections, a step removed from mere sensations. They represent the action of the neurones concerned in the child's manipulations, those concerned in his sensations and those concerned in connecting the latter with the former. They possess the satisfyingness of manipulation, of the love of sensory life *per se*, and something more, which, for lack of a better name, I shall call the *satisfyingness of mental control*. To do something and have something happen as the consequence is, other things being equal, instinctively satisfying, whatever be done and whatever be the consequent happening.
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> Now mental control, or doing something and having something happen,

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[1] "He [man] is an agent seeking in every act the accomplishment of some concrete, objective, impersonal end. By force of his being such an agent, he is possessed of a taste for effective work, and a distaste for futile effort. He has a sense of the merit of serviceability or efficiency and of the demerit of futility, waste, or incapacity. This aptitude or propensity may be called the instinct of workmanship." Veblen, *The Theory of the Leisure Class;* 1899, p. 15, quoted by Thorndike, *op. cit.* (p. 21), Vol. I, p. 143.

is satisfying in very many concrete forms. Not only making movements and thereby getting sensations, but also making an ideal plan and thereby getting a conclusion, making an imaginary person and thereby getting further imaginations of how he would act, and countless other "gettings and doings" are satisfying. They are originally satisfying since, as soon as training gives the ability to make the plan or image and get the result, nature gives satisfyingness to the connection.

67. Desire for excellence. — There seems to be a desire for excellence for its own sake quite apart from any consideration of its utility or the distinction which it brings. On the one hand, this desire is closely connected with pleasure in activity and on the other with the feeling of self-esteem that arises from successful effort. The Greeks said that "the gods see everywhere" and so made their workmanship on unseen parts as excellent as that on the visible parts. Very likely "the gods seeing everywhere" was an argument used to make acts appear reasonable which were performed for other quite real reasons. Even to-day some automobiles are made in which unseen parts are highly finished, and this is considered a meritorious quality, for which large sums are paid. Not a few people do excellent work just for the sake of doing excellent work; they say there is a satisfaction in it, and that ends all possible arguments.[1]

68. Success — self-esteem. — Not only does successful work create a feeling of pleasure while the work is being done, but after this feeling of actual pleasure in the work itself has passed away, the memory retains a notion of success which forms the basis for a feeling of self-esteem. The fundamental thing about all these matters probably is that whatever creates pleasure while the act is being performed or afterwards or possibly both will create a desire for the repetition of the act; that is, an interest will be created in that particular act. Many a time has the writer asked a young child what was the most interesting part of a day's school life and why, and practically invariably the answer showed that the interest was due to the fact that the child could do a certain act well and in some cases further questions brought out answers which meant that it was because it made him feel that he amounted to something. This, of course, persists in after life and is a universal feeling among mankind.[1]

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[1] " . . . The desire for excellence for its own sake graduates down from that of a Newton, or a Stradivarius, to that of the fisherman who, even when no one is looking and he is not in a hurry, delights in handling his craft well, and in the fact that she is well built and responds promptly to his guidance. . . . A large part of the demand for the most highly skilled professional services and the best work of the mechanical artisan arises from the delight that people have in the training of their own faculties, and in exercising them by the aid of the most delicately adjustable and responsive implements." Quoted by Thorndike, *op. cit.* (p. 21), Vol. I, p. 144 from Alfred Marshall, *Principles of Economics*, Vol. I, p. 147.

69. Curiosity. — There is a natural desire to learn more about that which one already knows in part. This desire may be sharpened by summarizing what is known and raising definite questions about what is not known. If the answers to the questions are delayed while their importance is made clear, they will be brought more definitely into the center of consciousness; the desire for the answers will be still more intensified; and when these latter are finally obtained, they are not likely to be forgotten. Curiosity in the hands of a judicious teacher is of great value as a motivating force.[2]

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[1] "One may say, however, that the normal provocative of self-feeling is one's actual success or failure, and the good or bad actual position one holds in the world. 'He put in his thumb and pulled out a plum, and said, "What a good boy am I!"' A man with a broadly extended empirical *Ego*, with powers that have uniformly brought him success, with place and wealth and friends and fame, is not likely to be visited by the morbid diffidences and doubts about himself which he had when he was a boy. 'Is not this great Babylon, which I have planted?' Whereas he who has made one blunder after another, and still lies in middle life among the failures at the foot of the hill, is liable to grow all sicklied o'er with self-distrust, and to shrink from trials with which his powers can really cope." James, *op. cit.* (p. 57), pp. 182, 183.

[2] "Curiosity is so valuable that, if man had not possessed it, he would never have explored his surroundings and might have been trapped and destroyed. If it were taken from man, one result would be that science would die, since curiosity is the strongest motive for scientific investigation." Charters, *op. cit.* (p. 123), p. 15.

"Intelligent curiosity, which is based upon partial knowledge, which reaches out to some definite end, and which leads to some adequate method of attainment of that end, cannot be overestimated as a means of development and training." Charters, *ibid.*, p. 178.

70. Histrionic interest. — The young child likes to "act things out." In "acting things out" there are coördinated activities of the mind and the body which afford simultaneously outlets for many and different tendencies to act. Moreover, this coördination of bodily and mental activities gives additional association-connections by which recall is made possible. Care must of course be taken that the main subject is not lost in the confusion that may result from the various possible physical acts. The thing to be learned must be given the center of the stage.

71. Derived or developed interest. — In the course of our lives we perform many acts which in themselves are of no interest to us or which are positively distasteful. This we are led to do through one or more of the many motivating forces which our fates have at their disposal. The uninteresting or distasteful act may be inseparably combined with other acts of sufficient interest to make a satisfying balance or they may be necessary to escape misfortune and pain or to secure what is desired either now or in the future. But as new acquaintances often lead to friendships and as forced association with those we are at first inclined to dislike may prove them possessed of admirable qualities which first disarm us and later enslave us in bonds of admiration and affection, so it happens that acts which circumstances cause us to perform, at first perhaps unwillingly, in the end engross our deepest and most abiding interest.

The young child has no interest whatever in learning to read. He knows nothing about it, and in the course of his evolution there has failed entirely to be implanted in him any spontaneous desire to find words on a printed page. He learns to read through the force of circumstances; it may be through the pressure of authority, the allurement of rewards, or a native desire to imitate, or a combination of all of these; learning to read may be so mixed with other highly desirable acts (motivated in the sense of section 56) that it appears as an inseparable part of an attractive whole. All these motives or interests are foreign to the reading itself, and only after considerable experience in reading does it acquire an interest on its own account — an interest which, in the case of most of us, is destined to grow and become one of the most abiding forces of life and one of the greatest sources of profit and enjoyment. The interest in reading is a *derived* or *developed* interest.[1]

This delineation of the development of interest in reading may be pursued much farther and represents fairly the growth of interests in general. The ability to read, once developed, leads to acquaintance with books of many kinds and to interest in the multitudinous subjects of which they treat — stories, history, biography, science — all the manifold fields of human thought and action.

One group of interests is constantly being superimposed upon others already acquired until the modern civilized man stands forth with his complex and manifold intellectual interests. There is thus a hierarchy of interests, nearly all acquired or derived, connected by a long and circuitous course of development (the details are interminable) with the comparatively few and simple interests which have their roots directly in man's inherited nature (see section 86).

72. Selection of interests. — Our environment serves to direct the course of development of interest by repressing some of them as they appear and by encouraging and developing others. The youngster is possessed of a multitude of potential interests, of a readiness to act in a large variety of ways. Some of these are discouraged upon their first manifestation, possibly never to reappear. Others are encouraged as soon as they show themselves, lead to pleasurable activity, and thus become firmly established among the child's permanent interests. The selection of certain interests out of a host of possible and budding interests is as certain and unerring as the selection of the so-called fittest by a particular environment. A clear recognition of this is of vital importance in developing those interests which are thought to be desirable.[1]

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[1] "*Involuntary intellectual attention* is immediate when we follow in thought a train of images exciting or interesting *per se;* derived, when the images are interesting only as a means to a remote end, or merely because they are associated with something which makes them dear." James, *op. cit.* (p. 57), p. 223.

Nor let it be supposed that this directing of the course of development of interests is a trivial matter. It is the tragedy of civilized life that as a race we have been thrust suddenly into a mode of life that is new upon this planet (new as periods of the evolution of the race go). This new life makes demands for which our course of evolution has not prepared our inherited natures and we are compelled to build for ourselves new interests, new powers of purposeful action; "Except a man be born again he can not see the kingdom of God." At this point, we quote Thorndike:[1]

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[1] "Put a child into a group of religious ascetics to grow up and the chances are that the only interests which will really get opportunity to live and thrive will be those which are conformable to the ideas of such a community. On the other hand, let him be cast among pirates, and a totally different group of interests will blossom forth. This is not because the child is a hypocrite. It is simply because one of the most universal of all objects of spontaneous attention is found in the attitudes and actions toward us of those among whom we live. A certain amount of repression from them may not stifle a vigorous interest. But many a taste, which might in a kinder social climate take root and bring forth rich fruit, dies ere it is fairly planted, because of the frosts of social disapprobation." Angell, *op. cit.* (p. 120), p. 424.

"Species of thoughts, like species of animals, prey upon one another in a struggle in which survival is the victor's reward. Further, just as species of animals fitted to one environment perish or become transformed when that environment changes, so mental forms fitted to infancy perish or are transformed in school life; mental forms fitted to one environment as school life perish in the environment of the workaday world; and so throughout the incessant changes of a mind's surroundings. The condition of a man's mind at any stage in its history is then, like the condition of the animal kingdom at any stage in the history of the world, the result not only of the new varieties that have appeared, but also of a natural selection working upon them. The tale of the human mind's progress is the tale of the extinction of its failures. Possibility of existence, stimuli to variations, selection by elimination: these words that describe the action of the environment on animal life are equally competent to tell the record of a human life." Thorndike, *op. cit.* (p. 21), Vol. III, p. 308.

> The original tendencies of man have not been right, are not right, and probably never will be right. By them alone few of the best wants in human life would have been felt, and fewer still satisfied. Nor would the crude, conflicting, perilous wants, which original nature so largely represents and serves, have had much more fulfilment. Original nature has achieved what goodness the world knows as a state achieves order, by killing, confining or reforming some of its elements. It progresses, not by *laissez faire*, but by changing the environment in which it operates and by renewedly changing itself in each generation. Man is now as civilized, rational, and humane as he is because man in the past has changed things into shapes more satisfying, and changed parts of his own nature into traits more satisfying to man as a whole. Man is thus eternally altering himself to suit himself. His nature is not right in his own eyes. Only one thing in it, indeed, is unreservedly good — the power to make it better. This power, the power of learning or modification in favor of the satisfying, the capacity represented by the law of effect, is the essential principle of reason and right in the world.
>
> Since original nature is neither all wrong, as our Puritanic ancestors tried to believe, nor all right, as the modern disciples of educational *laissez faire* try to believe, we cannot deal with it wholesale. Reason has to improve on nature without wasting it, by using each of its tendencies in view of all the rest and in view of the complicated apparatus of things and customs with which original nature interacts.

73. The child's present need of arithmetic. — Much has been written about "confining the activities of the school to that for which the child has immediate need or in which he has a natural and direct interest." If that could be done this chapter would be a description of the unnecessary. But it is never done, never has been done, except perhaps in rearing the children of the savage, and, let us hope, never will be done. In our endeavor to follow this dictum, we have broken all commandments on intellectual integrity and violated all canons of common sense.

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[1] *Op. cit.* (p. 21), Vol. I, pp. 281, 282.

The child of ten may have genuine need of all numbers, say up to twelve and of a few larger numbers. It would be interesting to find out how many children of ten or less ever actually use the combination $8 + 7 = 15$ outside the arithmetic class. Certainly no normal child of ten has a genuine need of $7 \times 8 = 56$, much less of finding the product of 87×354.

A few genuine child needs (not manufactured needs) for numbers can be found. Now and then children go to a store to buy things, but the number combinations required are usually simple. Some children's games require slight use of numbers; in the upper grades other school studies may require the use of simple arithmetic. All these and other child needs for numbers should be fully utilized for motivating work in arithmetic.[1]

It is of course easy to point out that one part of arithmetic is useful or necessary for other parts, but that does not show a "need" for arithmetic as a whole or for any of these parts. If the general dictum stated at the head of this section is given up, it becomes reasonable to use this interdependence as a motivating force. It is significant that at some point nearly all writers on this subject do give up this dictum and explicitly advocate a contrary procedure.[2]

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[1] "It is a matter of common occurrence for arithmetic to be used for its elemental purpose. A game is played. To record it and to select the victor, recourse must be made to notation and addition. A snowshoe is to be made of such and such a length, and denominate numbers are used. A sheet of paper is to be divided, and fractions occur. In all of these cases the intrinsic function is obvious and appealing. The purpose is seen and the need is felt." Charters, *op. cit.* (p. 123), p. 77.

[2] "Later on, in the fourth and succeeding years, the subject is, perhaps, studied in isolation, and the instrumental purpose of the subject matter for situations outside the subject is not so easily seen — perhaps cannot be seen when learned. For instance, it may be difficult to show why the multiplication tables should be studied for any other reason than that 'they should be.' And I think the best attitude to take is this: *You are studying arithmetic; you have done addition, but are slow. Let us shorten the process. The method of shortening is by multiplication, which does it in this way.*

"An ingenious teacher, however, can make concrete the need for multiplication by utilizing situations in which multiplication will actually save time, as, for instance, in keeping score where the teacher decreases the time for summation to such short duration that multiplication may be introduced as a heaven-sent device." Charters, *ibid.*, p. 78.

At the moment when this manuscript is receiving its final touches before going to the printer, there is before me a concrete example which shows how unnecessary to a full and normal child life is the arithmetic taught in the schools. A boy nearing his twelfth birthday, who has had less than one year of instruction in arithmetic (in the third grade) and practically no instruction outside the school, is completing the seventh-grade work in all other subjects and besides this has done two years' work with a high-school class in a modern language. His grades (including those in the high-school subject) are uniformly above the medium and his range of reading and general information is far beyond the average for his years. I doubt that he knows the multiplication table and I am certain he can not divide 8974 by 43. He has of course picked up a certain amount of information about numbers, including fractions, though he can not find the sum of 1/3 and 1/4. The point here is that his ignorance of arithmetic has not in the slightest cramped his development or retarded him in his other subjects, nor has he ever felt the need of learning more about arithmetic, and that in spite of the fact that he has for more than a year kept accurate records of all expenditures made directly on his account.

The simple truth is that, to save the formula that "arithmetic should be learned as a response to real needs of the child," we have deceived ourselves into thinking that such needs really exist. The fact is that no normal child has a real need of more than a very small fraction of the arithmetic which by common consent we regard as necessary in our school curricula.

74. Elements most effective in motivation in the lower grades. — The actual needs for numbers that arise in play or in other school work are *initially* the best motivating forces in so far as they are available. As a matter of actual practice, authority — the pressure of opinion on the part of parents, teachers, and others as to what children should do — is a powerful motivating force. Among the other child tendencies, imitation, group instinct, group rivalry, pleasure of activity, and feelings of success and self-esteem are among the most effective motivating forces in the early grades. We now proceed to a discussion of various methods of connecting these interests with the work that we desire the child to do, thus creating composite situations which will be attractive and through which we may reasonably hope to create a derived interest in the work itself.

75. Games of rivalry. — The games which are most generally used in teaching arithmetic and which are also by far the most successful are games of rivalry.[1] Less frequently the rivalry is personal and more frequently between groups. During the last decade or longer, our teachers' journals have contained descriptions of many number games which are supposed to be suited for use in the lower grades. On studying these games one is impressed with the fact that they have not been effectively scrutinized with a view to the selection of those which possess the qualities that go to make up a good game of rivalry. The following characteristics should be possessed by a good number game:

a. It should keep all the children at work during practically the whole period devoted to the game.
b. It should not lead to noise or confusion which distracts the attention from the subject.
c. It should create in the child a keen desire to learn.
d. It should make the element of group rivalry stand out sharply.
e. It should make the group exert pressure on the individual to do his best work.

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[1] "The principal device currently used by teachers in fixing the fundamental arithmetical concepts is the group play of children. Games permit of group work where the attention is predominatingly spontaneous." Brown and Coffman, *op. cit.* (p. 31), p. 140.

"Various social instincts can be utilized in matches after the pattern of the spelling match, contests between rows, certain number games, and the like." E. L. Thorndike, *op. cit.* (p. 42), p. 195.

f. It should make each pupil vitally interested in the work of the other members of his group.

The purpose of such a game is to cause the child to work at his highest pitch throughout the period and to do this with a vigorous desire to improve both in speed and accuracy. He should be caused to do this joyfully; and when the game is over, he should be anxious to try it again and should be possessed of a firm determination to do better the next time. To attain these ends it is important that the teacher should understand the proper qualities of good number games and the details of the purpose for which they are used.

In many of the games proposed only one or two pupils are at work while the others watch in idleness. Each pupil is occupied in number work, say one or two minutes out of a ten- or twelve-minute period devoted to the game. Aside from the waste of time, such games are vicious because attention drifts to extraneous things, and it will be a miracle if some disturbance is not created which will take the attention of the whole class entirely away from arithmetic.

It is, of course, clear that a good game must not create noise or confusion. Noise and confusion in a class are bad, not because children should not be noisy or act in an unregulated manner in general, but because noise and confusion take the attention away from the subject matter, and whatever does that is fundamentally bad.

If the game does not make the element of rivalry stand out clearly, it fails to make use of the chief element of motivation which is the reason for its existence. Group rivalry is perhaps the one most important device for motivating elementary work.

If a game is so arranged that the members of a group begin to exert pressure on the individual, use will be made of one of the strongest elements in motivation. Children, like older people, are exceedingly anxious to secure and retain the respect and good will of the members of their groups. Suppose that the failure of one child to know his number combinations causes his side to lose. There will then be a definite public disapproval of his conduct, and we all know that it requires a strong character to be even fairly independent of public approval or disapproval. Whether or not that child likes to study the multiplication table or the table of addition combinations, he will be so anxious to avoid the disapproval of the other members of his group that he will go to his work without even a thought that it is not agreeable to him.

It is but a step from this condition of affairs to the point where the stronger members of the group begin to take an active interest in their fellows. To secure this result the same teams should play together over a considerable period to permit team rivalry to develop. In one school where such games were used effectively, little Johnny came to his teacher with tears in his eyes, saying that Harry had pummelled him. "What for?" asked the teacher. "'Cause I didn't know my combinations and our team got licked." In this school it was not unusual to see the brighter boys and girls drilling the duller and slower. The keenest interest was taken in the progress of these slower ones, and every means which the youngsters could think of to help them along was being used. It is hard to overestimate the influence of the social pressure which thus was brought to bear on each one to do his utmost and also of the salutary interest which each one was made to take in his fellows.[1]

76. Examples of bad games. — The following game is taken from a recent textbook on arithmetic by a distinguished author. "Let four boys hide behind the teacher's desk; then let six more boys hide behind the teacher's desk, and then let the rest of the class say how many boys are hiding behind the teacher's desk altogether." It does not require an experienced teacher to see how much commotion and disorder and distraction of interest there would be to the square inch of arithmetic.

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[1] See note on page 137.

Following is the description of another game taken from a recent text. One child stands before his class and says, "I am thinking of two numbers whose sum is eleven," or some other number that he has in mind, and the other children take turns asking questions something like these: "Is it four and seven are eleven?" "Is it eight and three are eleven?" and so on until the combination of which he is thinking is named. The child who is the first to name the desired combination stands before the class and gives a new sum whose components are to be guessed. The bad elements in this game are apparent enough though not nearly so glaring as in the one described above. What are the other children doing while one child is asking questions? The most that any one child can do will be to think of a different combination as "five and six are eleven" and then to keep that combination in mind until he is called upon to ask his question. On an average, each child will wait a considerable time before his turn comes and the result is that he does about one fifth or possibly only one tenth as much work as he might do in the time given to the game. A game of this sort may be satisfactory as a diversion for very elementary classes, but it is very far from possessing all the elements which are essential in a good number game.

77. Examples of good games of rivalry. — A class is divided into two groups. The groups may be given names, possibly the names of two neighborhoods or of two baseball teams or of two schools in which the children are interested. The teacher shows the class cards each containing two numbers to be added. As a card is shown the children look at it and write down the sum. After a number of cards, such as ten, have been shown, the children exchange papers in such a way that no one has a paper belonging to a member of his own team. The teacher then reads the correct results in the order in which the combinations were given to the pupils. Very rapidly the number of mistakes in each row is found (for convenience in scoring, all pupils in a row belong to the same team) and written on the board, one column for each team. These columns are added to find the total number of mistakes for each team. This very simple game possesses all the good elements that were enumerated in section 75. After the game has been finished, the teacher may suggest that possibly the pupils would like to play the game over again in the future. With the least possible suggestion from the teacher it is finally decided that the same game is to be played over again soon and that the same teams are to play. If the same teams play together for several games, all the elements which we have described above will be brought into play. Later, the game may be varied by choosing two captains and letting them select their teams. If they are fairly evenly matched in point of numbers, the girls may play against the boys, or one side of the room may play against the other side.

A game called a "field meet" is among the best: The class is divided into several teams, say four. The captains are selected by the teacher or by vote of the pupils and the captains then choose the teams. If the captains are numbered from one to four, captain number one chooses a player, then captain number two chooses one, etc. Captain number four chooses two players, when his turn comes. That is, he gets the fourth and also the fifth choice. Captain number three gets the sixth choice, etc. Thus the choosing goes back and forth until all members of the class have been chosen.[1] A ten minute test, say in addition, is now given, the papers graded,[1] and the total score for each team is found. This test is the first event. Sometime later, possibly the next day, another event is run off in which a different operation may be used. When a number of events have been run off, the field meet is completed and the total scores posted. The team with the highest score wins. In each test enough work should be given so that the fastest worker will fail to do it all. The players should be started and stopped promptly.

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[1] The captains are very likely to know who are the best "players" and the best ones will be chosen first. By the method suggested here the advantage of having the first chance is much minimized. Suppose there are twenty children and five teams. Then the captain of team No. 1 has the 1st, 10th, 11th, and 20th choice while the captain of team No. 5 has

The game called a "cross-country run" is perhaps even more effective: Captains and teams are selected in the same manner as for the "field meet." A test is given containing an amount of work which can be finished by the fastest worker in about half the time to be used for the test. As each pupil finishes, he raises his hand and turns down his paper. The pupil who finishes first, gets *one* as his score, the one who finishes second gets *two*, etc. If two pupils raise their hands at the same time, they finish a tie and are given the same score. When half or two thirds of the class have finished, all the others are given the same score and stop working. In grading the papers a certain amount, say 5, is added to the score for each example that is wrong, not finished, or not attempted at all. The total score for each team is found and the team with the smallest score wins.[2]

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the 5th, 6th, 15th, and 16th choices. By the usual method the captain of team No. 1 would have the 1st, 6th, 11th, and 16th choices while the captain of team No. 5 would have the 5th, 10th, 15th, and 20th.

[1] The papers may be graded in a minute or two by having the pupils exchange papers and mark correct results as the teacher reads them. (The papers should be so exchanged that no pupil scores a paper belonging to a member of his own team.) The pupils read the grades of the papers in their possession while the teacher enters them in the proper columns on the board. After the scores are computed the papers may be returned to the owners when objections to the grades may be raised. Anyone scoring a paper incorrectly should have a certain number of points deducted from his grade. Exercises should be chosen whose answers are already in the teacher's possession, since this leaves the teacher free to give general supervision to the game.

[2] The papers may be scored as in the preceding game. Adding a certain number of points to the score of each example finished but with incorrect answer puts a high premium on accuracy while the effort to come in early puts a premium on speed.

These games possess all the good qualities enumerated in section 75, with the exception that in the cross-country run the fastest workers are idle part of the time. This, however, is not serious since the idle time, except for a very few, is insignificant. They put a high premium on both speed and accuracy and put full pressure on the fastest as well as on the slowest. Games of this kind may be used to advantage much more extensively than is usually the case at present. For more detailed suggestions for the use of such games, see the Teachers' Edition of the "Applied Arithmetics" by Lennes and Jenkins.

Proper variations of such games with an eye to the fullest development of group competition will go a long way in removing the feeling of drudgery so frequently associated with drills, as well as in removing every just basis for the charge of ineffectiveness in developing reasonable speed and accuracy in the fundamentals now so often made against our elementary schools (see section 93).

The objection is sometimes raised against games of this kind that, while they stimulate the better students, they confuse the slower and actually retard their progress. This objection is not believed to be well founded in the case of the games suggested here. The slow students have the same things at stake as the faster ones. Each is trying his best to come in ahead of some one. Moreover during the race no student knows his position in the race with respect to the others. For the sake of the team, it is just as important for the fastest student to work up to his limit of speed and accuracy as it is for the slowest.

78. The ethical quality of rivalry. — Much has been said about the ethical quality of individual rivalry as a means for motivating school work. It is contended that it will make some individuals stand out as definitely superior to others and that this will have unfortunate effects both on the superior and the inferior, the former tending to become afflicted with undue self-esteem and arrogance while the latter become discouraged and possibly morose. This may be true, at least in exceptional cases, but personal inequality is one of the fundamental facts of life to which all must become accustomed sooner or later. The surest way to minimize this difference in the intellectual domain is to abandon education, for training increases intellectual differences instead of decreasing them.[1] Since we are greatly different in our intellectual endowments, it may not be amiss to let this fact stand out early in life (it is sure to do so later unless we abandon civilized life with its endless complexities and excessive demands upon intellectual ability) and to apply at the same time whatever means may be possible to make all reasonably happy, the one with his one talent and the other with his ten. When all is said and done, it remains true that the desire for personal distinction is one of the most universal and most potent of human motives, and the teacher may be pardoned if now and then use is made of it in motivating school work.[2]

However, the games suggested above, as well as the great majority of those in general use, appeal to rivalry between groups rather than between individuals, and such rivalry should serve to develop some of the finest elements in human nature. Indeed it would appear that none of the arguments that have been urged against individual rivalry, justly or unjustly, can be urged against group rivalry: Willingness to do teamwork, to play hard with one's group, to feel keenly personal responsibility to it, to take a helpful interest in one's team mates, and to learn to appreciate the contributions of other members to collective success — these are some of the most important lessons to be learned, be it for school or for later life.

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[1] "The facts found are rather startling. Equalizing practice *seems to increase differences*. The superior man seems to have got his present superiority by his own nature rather than by superior advantages of the past, since, during a period of equal advantages for all, he increases his lead." Thorndike, *op. cit.* (p. 21), Vol. III, p. 305.

[2] "But gradually, the work becomes more reflex, 'speeding-up' exercises should be given. Here is a place for the right kind of emulation, such as is found in contests among classes, or among individuals. There is a tendency among the advocates of 'soft pedagogy' to disparage rivalry in the schoolroom. History shows us, however, that this motive has been a powerful factor in every line of social and individual progress. Because rivalry has a selfish, antisocial side, it does not follow that it lacks a noble and helpful one. It is not well to foster emulation to the extent done by the Jesuits, who went so far as to pair off all the boys of a school, making the individuals of each pair rivals in everything pertaining to school work. It can be used safely, however, in pitting class against class or, if tactfully done, individual against his fellows of the same class. Within these limits emulation will prove itself a powerful schoolroom incentive." Brown and Coffman, *op. cit.* (p. 31), p. 158.

It is difficult to overestimate the driving force of group rivalry. Those great practical psychologists, our industrial leaders, know this well enough. When Schwab, during the war, wanted his crews to rivet ships as they were never riveted before he played off crew against crew. Records were posted and every good crew in the shipyards tried to break them. When a regiment was asked to go over the top in France, what were the motives that were urged to make the men do maximum execution in the face of death? Was it patriotism, or love of home and fireside and mother and sweetheart? Was it appeal to the flag, the emblem of the nation and its institutions? When the supreme test came, it was none of these. "The Seventy-second Regiment never turned back" —that was it. Loyalty to the group (not a group so large as to be too vague), loyalty to the gang of buddies, pride in the record of the Regiment — that was it.

It is interesting how an individual transferred from one group to another identifies himself with the new group. This shows that the motivating force lies not in working with a *particular* group but in working in a *group in competition with other groups.* A riveter in the shipyards would work with high and good-natured glee but with might and main trying to outdo another gang. If after a while he was transferred to another crew, he would soon work just as hard and take just as much interest in making new records as he did with the former gang. Group rivalry is morally one of the cleanest of motives and practically one of the most effective means of getting people, be they young or old, to do really hard work and to do it with cheer and with a minimum feeling of fatigue. Responsibility to the group, to the opinions of the neighborhood, and the desire for good repute among one's fellows are with us to the end and serve as nothing else to keep us in the right paths.[1]

79. Records of individual tests. — At stated intervals of time, say once a week, a test may be given, the result of which is to be entered on a permanent record card. Part of such a card is shown on the next page. There should be two cards for each pupil, one to be kept by the teacher and one by the pupil.[2]

If it is at all possible to secure standard tests, this should by all means be done since that enables the child to compare his record with standard achievements. The time for each test should be so regulated that a pupil working at standard speed will do half of the work.[3] In case the teacher makes up the tests she may request each child to hold up his hand when he has completed half the work. She may stop the work when about half of the pupils have held up their hands. This latter method will serve simply to compare the members of the class among themselves and will afford no indication of the achievement of the class as compared with other classes. However, even this kind of tests focuses

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[1] "The man may be neither better nor worse off on account of the opinion held of him by society, but innate tendencies give the opinion of society a force that he cannot overcome. These influences hold the man in his accepted place, keep him to his allotted task in moments of weariness, prevent eccentric acts and remarks, are the forces that make society possible, even if in many cases they make convention dominate originality. In the individual they serve as spurs to many of the activities with a more remote purpose, they give the ideal and unselfish aim an approximation to equal standing with the material and the selfish." Pillsbury, *op. cit.* (p. 126), p. 434.

[2] "The interest in achievement, in success, — is stronger in children than is often realized. . . . Children who thus compete with their own past records, keeping an exact score from week to week make notable progress and enjoy hard work in making it." Thorndike, *op. cit.* (p. 42), p. 197.

[3] Lennes and Jenkins, *Applied Arithmetic*, J. B. Lippincott Co., 1920, Teachers' Edition, Book 2, pp. 308, 309.

Name	Grade															
Month	September			October					November				December			January
Date	11	18	25	2	9	16	23	30	6	13	20	27	4	11	18	
Subject	Add. Int.	Sub. Int.	Mult. Int.	Div. Int.												
100																
90																
80										x						
70				x		x	x		x		x					
60	x		x		x			x								
50		x														
40																
30																
20																
10																
0																

Part of Pupil's Record Card

the child's attention upon his own progress rather than upon his comparative success.

After a few months' work has been recorded the general quality of a pupil's work becomes apparent and the child will then be inspired to try to raise his grades nearer to the top of the card. If such tests and records are persisted in throughout the year and from year to year, they will form very valuable stimuli. A desire to break one's past record is one of the strong forces of life.[1]

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[1] "To work for marks is not intrinsically bad. If the marks are, as they should be, correct measures of either the amount of knowledge, power, appreciation, and skill attained or the amount of progress made, to work for marks means simply to work for knowledge, power, increase in knowledge and power, and the like as recognized and measured. . . .

80. The pleasure of activity. — There is a positive pleasure in an activity for which the individual is well adapted. The dog likes to run and catch things that are thrown or to run after moving objects and bark and snatch at them. It is believed that, during the long ages of the dog's evolution when he obtained his food by catching and killing, his anatomy became so perfectly coördinated for performing these acts that now grabbing and shaking (make-believe for catching and killing) are quite the most enjoyable things that a dog can do. The enjoyableness of an act does not depend altogether upon inherited adaptation for performing the act but upon one's present adaptation for it. There is inherited adaptation for running and jumping and yelling and these are learned without urging by teacher or parent, but there would seem to be no such inherited adaptation for skating or riding a bicycle or playing the piano or singing an aria. That is, the inheritance is not such as to lead inevitably to the doing to these acts, though it is sufficiently plastic to make the learning of these acts possible. After one has learned to skate or ride a bicycle or play the piano or sing an aria so that these acts can be performed easily and effectively there is a positive joy in doing them. That man is most perfectly adapted to his life work who likes to perform the acts required of him as the dog likes to run and catch the thrown stick. As we have just

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"The proper remedy is not to eliminate all stimulus to rivalry, and along with it a large part of the stimulus to achievement in general, but to redirect the rivalry into the tendencies to go higher on an objective scale for absolute achievement, to surpass one's past performance, to get into what, in athletic parlance, is called a 'higher class,' to compete within that class, and to compete coöperatively as one of a group in rivalry with another group. . . . .

"To be seventeenth instead of eighteenth, or twenty-third instead of twenty-fifth, does not approach in moving force the zeal to beat one's own record, to see one's practice curve rise week by week, and to get up to the standard which permits one to advance to a new feat. Mr. T. H. Kirby found in the case of fifth-grade pupils that, by thus reporting to each pupil his absolute achievement in measured tests in addition, sixty minutes of drill resulted in an improvement of over 50 per cent. in speed with a slight gain in accuracy as well." Thorndike, *op. cit.* (p. 21), Vol. I, p. 289.

seen, this liking can be developed — is very likely to develop as the organism becomes adapted to its task through practice. The teacher must be on the constant watch for this "love of the work for its own sake," and her aim should be to develop such efficiency in at least the fundamentals of arithmetic that the child will like to practice on them.

REVIEW QUESTIONS

1. Discuss the usefulness of arithmetic as a motivating force in the lower grades. To what extent has the young child real natural needs for arithmetic? (§§ 52, 73.)

2. Describe the nature of interest. (§ 53.)

3. Give examples of motivated acts interest in which is derived not from the acts themselves but from elements closely associated with them in the immediate present. Also give examples of motivated acts the interest in which is derived from remote considerations. Which of these types of motives is more effective with the child? (§§ 54, 55.)

4. Discuss the definition of motivation given in section 56.

5. Discuss the child's spontaneous interests as given in sections 57 to 70 of this book. In particular consider personal distinction and rivalry in contrast with group rivalry. (§§ 59, 61, 62.) Contrast desire for personal distinction and desire for excellence. (§§ 59, 67.)

6. Consider the best means of making simultaneous use of interest in what is already known and interest in what is new. (§§ 63, 64.)

7. What is meant by "derived, or developed, interest"? Discuss such interests as an important aim of education. (§ 71.)

8. Discuss the selection of interests and its importance in education. (§ 72.)

9. Discuss the most effective means of motivation in the early grades. (§ 74.)

10. Enumerate the elements of a good game of rivalry for school use. What elements of natural interest does such a game bring into use? Give examples of bad games with reasons why they are bad. (§§ 75, 76, 77.)

11. Consider the ethical quality of rivalry, (a) of individual rivalry; (b) of group rivalry. Discuss the rôle of individual rivalry in the lives of adults. (§ 78.)

12. Discuss systematic individual records as a motivating force. (§ 79.)

13. Discuss pleasure of activity, (a) as a motivating force; (b) as an aim of education. (§ 66.)

CHAPTER IV

MOTIVATION IN THE GRAMMAR GRADES

81. Continuation of the means of motivation used in the elementary grades. — Many of the motivating forces to which the child responds in the elementary grades continue in effectiveness in the upper grades and indeed through life. Some of them become even more effective with increasing age. Curiosity is developed by discovery; one who has found part of what lies beyond has his appetite sharpened for more. Pleasure in successful work grows with age; the instinct of workmanship — the desire to make one's own plans and to carry them out till they become objective realities — grows stronger as the years go by. The force of both personal rivalry and group rivalry becomes more compelling as one grows older.

The games of rivalry described in section 75 may be continued in the upper grades. Not more than a generation ago, large use was made of the element of team rivalry even among mature people. The old-fashioned country spelling matches undoubtedly made those who took part in them better spellers than they would have been otherwise. The group contests threw about the tedious task of learning to spell an atmosphere of cheer and good feeling which was in strange contrast to the struggles which farmers are now making to learn to keep farm accounts — a struggle which is motivated by all the assertions that extension workers and county agents can make that keeping accounts will pay the farmer a greater return in dollars per hour than any other work he can do and motivated too by the necessity of keeping such accounts as will satisfy the internal revenue collector in regard to taxable income.

82. Increasing importance of derived interest. — Derived interest becomes increasingly important as one grows to intellectual maturity (see section 55). If the subject of arithmetic is properly taught, the pleasure which accompanies successful activity should be increasing throughout the upper grades. When a child in the fifth grade says that he hates arithmetic, it is likely to be because he has been unsuccessful in performing the tasks which have been assigned to him, or possibly because they have been so easy as to be uninteresting. If, on the other hand, he has been comparatively successful in dealing with the tasks assigned and, if at the same time he found them difficult enough not to be monotonous, the joy of activity will have made the whole subject pleasant for him. A feeling of personal achievement and effectiveness will contribute to this pleasurable feeling, and the pupil will have gone a long way toward liking the subject for the pleasure which it affords.

It is important that the subject should not be distasteful at the outset. Those who train colts, especially if they are of high grade, are careful not to annoy them or scare them in the early part of their training. A similar method should be adopted with the young child. The development of industry — of the perseverance which belongs to the effective mature man — must come gradually as all things in nature do.

Natura non facit saltum — "Nature never makes a leap." [1] But nature does move forward; growth is her law. It is by this gradual process, made agreeable at the outset by whatever means may be necessary, that new hierarchies of interests develop most rapidly and form the strongest growth. In steering clear equally of the initially distasteful on the one hand and the too easy tasks of a soft pedagogy on the other lies much of the secret that leads to the fullest development of the child's inherited power and the greatest interest in his work.

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[1] There are no "mutations" in individual growth.

83. The more mature type of imitation. — As the child grows older, he approaches more closely the intellectual status of adults. He no longer imitates only the simple movements and expressions of those about him, but takes on more and more the point of view of older persons. He begins to admire the capable and successful and those possessing strong and effective personalities. If the teacher is of the right sort, this tendency in the child may be used to motivate his work. Sermons preached by the teacher have little effect, but the approval or disapproval of one who is highly respected may have great effect.

The point here is that, as the child grows older, he has an increasing tendency to *select* that which he will imitate. It is at this point that ideals come into play. Now, ideals, especially among the young, are general rather than specific. The ideal of the child is not of a person clever in doing a particular act but of one who is clever and masterful in all relations of life. There is a strong tendency to find this ideal in a person and not in a body of abstract virtues.

It is not truthfulness, bravery, patriotism, intelligence, industry that form the guiding star of the young, but rather some *person* possessing these qualities. It is this fact which makes the teacher's responsibility great. The pupils are more likely to follow her example than her precepts provided she is strong enough to become the personal embodiment of what the child regards as worthy; and if she is not, they are very likely to follow neither her example nor her precepts.

At this stage the building of right ideals is important, and in this work all teachers may coöperate. If there is anything that carries over completely from one subject to another, it is our *ideals* (not merely *practice*) of, for instance, effectiveness and industry.

84. Interest in human affairs. — As the child grows in maturity, he acquires a deepening interest in human affairs and especially in those outwardly visible affairs which constitute industry and business. He is more interested in facts than in the principles which control them. Effective use can be made of this interest in motivating arithmetic. The difficulty in much of the work in the upper grades is not one of actual arithmetic but of information about the social and economic bases which give rise to the problems.

The subjects of interest, stocks and bonds, discount, investments, taxes, etc. are all very simple so far as the actual arithmetic content is concerned. The real difficulty is that the words used are often little more than mysterious symbols and that the rules which are developed and committed to memory are little more than successions of sounds which carry no meaning. It is here that the child's interest in human affairs may be utilized. If the child understands that the problems he is asked to solve are based directly on what is going on in the stores that he visits, in the banks which he passes daily, and in other places where people are working and transacting business, he will thereby be led to acquire the information necessary to cope intelligently with these more advanced topics in arithmetic.

There is, of course, another phase to this whole subject; namely, that it is the business of a course in arithmetic to furnish information about those facts of life which are most intimately connected with arithmetic and which all reasonably well-educated people should know. The specific information which should be furnished in connection with each topic will be considered when the teaching of these topics is discussed.

85. Rendering work agreeable. — One point must be added to those noted in the preceding sections: Not only must work be motivated in the sense just described, but each individual task should be rendered as agreeable as possible by any means at one's command. No matter how set one's purpose may be to achieve a large and worthy aim, it will be achieved more quickly and more certainly if each necessary task is itself agreeable. This is most important for the young child, but it is very important even for the mature. When we are devising games and other means of rendering the task agreeable for children in the grades, let us not think that we are dealing with a matter that should be restricted to childhood or is in fact so restricted. The young men who uphold our athletic honors in college plead with us to come out and cheer. They know that it will help them ignore the pain that comes from extreme exertion and make them more likely to win.

The wise doctor who knows that his patient must have exercise inveigles him into playing some game, for he knows that unless the exercise is made agreeable his patient, mature and experienced as he is, will not take it though the prolongation of life depends upon it. To experienced workers who have studied the matter of personal efficiency, this is an old story. Some have more than one piece of work going at the same time. When one grows tiresome, another may prove inviting. As the work proceeds, one piece of work may become absorbing and hold the interest for a long time, in which case free rein is given to the inclination. Some work better in collaboration with others when the interests may be shared and feelings of aversion prevented by comradeship. Personal comfort while the work is being done, interruption of work only at the will of the worker, prospects of agreeable diversion when the work is finished will increase the efficiency of the worker.

The prospect of a game of golf at the end of the afternoon's work makes otherwise tedious labor more agreeable and a planned fishing trip at the end of the week casts its beneficent glow over the troublous tasks of the week and makes them easier of accomplishment. The negro who said, "To-morrow is Thursday, the next day is Friday, and the next day is Saturday night" was a profound philosopher. We have said often enough that a firm purpose to achieve a worthy aim helps greatly in making the required tasks agreeable; we are now trying to emphasize the importance of making these tasks still more agreeable by all possible means. Too often the task has been made agreeable by making it so simple that it will require no effort. Our task is a far more difficult one — to make exertion to the utmost as agreeable as possible.[1]

86. Motives of the adult: transient and permanent motives. — It will be illuminating to consider a little more in detail the motives that actuate the adult. We have already contrasted the tramp who saws wood because it is the only means of satisfying the immediate cravings of hunger and the farmer who saws wood because it will be needed a year hence. The tramp works only under the pressure of immediate necessity and quits as soon as this pressure is relieved; his motives are transient, and his tragedy is their unfailing recurrence.

The farmer's motive for sawing wood for which there is no immediate need arises from a general purpose of securing for himself and his family reasonable economic welfare. This purpose is in part due to fear of poverty and its attendant misery and in part to the pressure of public opinion. The child acquires from his family general ideas about propriety of conduct, about economic respectability, and general social standards. The responsibilities of the adult transform these ideas into general ideals of life, which are the bases of our permanent motives. So it is that man learns to "look before and after." Each act is judged good or bad as it is in accord or discord with his permanent motives; the day's tasks are done, and he lives a life of purposeful work against which poets and "choice spirits" rebel.

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[1] "There is no other means of arousing zeal for a given course of thought or conduct than by connecting satisfaction with it; the mind does not do something for nothing." Thorndike, *op. cit.* (p. 21), Vol. I, p. 295.

"Most important of all cases of this process is the shifting of satisfyingness and annoyingness. . . . Satisfyingness and annoyingness may, under the limiting condition noted above, be attached to any situation whatever. So, unhappily, man may come to be made wretched by simple outdoor sports, children's merriment, spectacles of cheerful courage, or the daily panorama of sensory experience. So, to his very great gain, man may come to welcome productive labor, excellence for its own sake, consistency and verification in thought, or the symbols of welfare in men whose faces he can never see." Thorndike, *ibid.*, Vol. II, p. 31.

But we have yet to consider the motives of those tireless workers whose purposes seem to contain no "miserable aims that end with self," of those "whose thoughts sublime, pierce the night like stars, and with their mild persistence urge men's search to vaster issues." Of the countless number of such men Charles Darwin is a glorious example. The heir of sufficient wealth to provide for all his material wants, Darwin probably never did a piece of work for the purpose of obtaining material reward. Interest in the natural sciences developed early, and on his voyage on the *Beagle* he conceived the theory of the formation of species by natural selection. Fully aware of the far-reaching, if not revolutionary, character of this theory, he labored for twenty years to prove or disprove it, until in 1857 circumstances forced the publication of "On the Origin of Species." Not content with having produced what has generally come to be regarded as the greatest book of his century, he continued to labor unceasingly and with remarkable effectiveness until death called him twenty-five years later. What were his motives? No doubt they were many and complex. Scientific curiosity may be thought by some to account for it all, and no doubt it played a great, perhaps the chief, part; but who shall say that it was not supplemented by other motives so that he was led to greater achievements than would have been caused by it alone? A desire that the world might be richer because he had lived (not merely in a financial sense, though he probably contributed more to the world's material wealth than any captain of industry or business in his century); distinction for himself, his family, and possibly for his nation; the approval and admiration of his intimates and fellow workers; a vivid scientific imagination, which crowded his vision with more problems than he could possibly solve and with more avenues of attack than he could ever attempt; the consciousness that what he was attempting to do was worthy in the highest sense — all these no doubt combined to form a life motive which completely dominated him and which made him one of the most useful and effective of the great workers who have left the legacy of their achievements to a struggling world.

In the growth of permanent life motives there is a struggle for survival among many discordant motives. The selection of the motives that ultimately become dominant is controlled by many forces. Inherited characteristics, early encouragement of some tendencies and discouragement of others, the formation of ideals, pressure of social approval and disapproval — all these have their shares. This is a struggle which never ceases. Our most permanent motives are never left in undisturbed possession of the field. The tragedy of life is that we are endowed by inheritance with hosts of inclinations that run counter to the plans of life which are approved by our calmest and most abiding judgments. Our transient motives are constantly disputing the mastery of those that are more permanent.

> "I could easier tell twenty what were good to do
> Than be one of the twenty to follow mine own instructions."

Difference in "character" consists largely in this, that in some the transient motives temporarily eclipse those that are more permanent causing acts that do not receive the approval of poised judgment, while in others the permanent motives are never lost from view and their acts remain consonant with well-considered purposes of life.

87. Selection of motives. — In the schools much may be done to develop the higher and more permanent types of motives and this end should be kept constantly in view in selecting means of motivation. The higher motives of the adult make no appeal whatever to the child. Fear of punishment, hope of immediate reward, a tendency to personal rivalry — these the child shares with us all; but he starts school life with no far-reaching purposes and with no desire for that which is useful later on. It is agreed that fear of punishment is not an ideal motive; partly, because it is repugnant to our humanitarian feelings; and partly, because it alone can never secure the best results.[1] Rewards and distinctions for good work are entirely legitimate; most of the world's work is done for some kind of reward, be the worker young or old. The essential difference between rewards offered in the schools and the higher rewards of life (this includes the largest rewards paid in the money of the realm) is that the school rewards are immediate while the rewards of life are often distant.[2] Hence it is necessary at the outset to use many semiartificial motives, such as are described in Chapter III.

As soon as possible a beginning should be made in developing ideals of life and consequent purposes to attain them. Ambition to achieve, to excel one's own record, to excel standard records, and even to excel the records of competitors is altogether praiseworthy. Of late we have not set as much value on personal ambition as it deserves. The early development of ideals of life does not imply early selection of vocation or career, but it does imply the early development of a strong purpose to make life splendid in personal qualities and in achievement. "Such a person as you want to be can not be ignorant of this and of that."

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[1] In some cases it may be necessary to resort to actual punishment, and this is much better than to let a willful child have his way. Now and then it will be found necessary to use the means which nature herself has employed to force obedience from her children. Starvation or destruction by enemies are the punishments that she inflicts. "The devil take the hindmost" is her law, and she enforces it pitilessly. Yes, the rod is an ancient institution and is still necessary at times. But in general it is not the most effective motivation to action. Man has improved on nature here as in many other respects (who will deny that the million contrivances devised by man for his own comfort are an improvement on the "natural" environment of the naked Patagonian?) and the rod is to be used only when all else fails.

[2] One young engineer works for a certain salary and does nothing except to earn that salary; another works for the same salary but does twice as much work in the hope and expectation that twenty-five years later he will be distinguished in his profession.

The use of artificial means for making work agreeable is entirely consistent with the highest type of motivation. A boy who has decided with all the earnestness of his nature that he wants to learn arithmetic because without it he can not be the kind of a man he wants to be may be stimulated to much more effective work by games of rivalry or by the promise of something attractive when the work is done; and such stimulus is entirely legitimate and in no wise does it tend to weaken the permanent motive. The wiser among us grown-ups are constantly doing the same thing. The best worker is he who so orders his life that its controlling motives are well considered and its daily tasks performed under the most attractive conditions possible.[1]

But when all is said and done, we must beware of the idealistic theorists who deprecate all "artificial" means of motivation and who would seek to rely solely upon the efficacy of intrinsic interest and "love of the work." Not long since, the writer was berated soundly for holding the ideas on motivation expressed in these chapters. "Motives such as you describe have no place in a modern school." A little later I elicited from my uncompromising critic (a doctor of philosophy in education and a teacher in a well-known school of education) that if it were not for economic necessity she would spend her winters in Florida or California and her summers in the mountains and give the schoolroom a wide berth forevermore. "Love of the work" might be sufficient to hold little restless tots to their prosaic tasks, but not the maiden of forty. If it were not for this same economic necessity, there would be a shortage of teachers the like of which has never entered our dreams. If the most distinguished of educators were to receive the same income and in all respects retain their preeminence and the same high respect of their fellows irrespective of whether they continued their teaching and investigations or not, we might look for an unprecedented number of vacancies in our great universities. Only in rare instances, among children exceedingly rare, is "love of the work" sufficient. Necessity (in the case of children, authority), personal prestige, group fidelity, such are the motivating forces that are necessary to keep the majority of us at our tasks. Love of the work is important, but it is not sufficient.

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[1] "*School incentives* are generic values. The desire to be promoted, to make a grade, to be first to secure prizes and rewards, to avoid corporal punishment, to please the teacher, to escape detention, and to live up to class opinion are all motives widely if not wisely used. These motives have to be selected with the greatest care, because being school motives they do not operate in quite the same form in adult life, and, if pupils form the habit of relying upon them as a spur to action in school, they may have no spur to take their place in performing the drudgery and prosaic details of life.

"Pupils differ at different ages in the motives that appeal most efficiently. Love of play is a good motive of one age, a poor motive later. Duty is an adult motive but not one that appeals to children, especially boys. Future success appeals to the adolescent, but not to his younger brother." Charters, *op. cit.* (p. 123), p. 168.

88. The value of motivation. — It is a uniform result of common experience and of carefully performed experiments in the laboratory that one learns much faster when one has a keen desire to learn.[1] It is common experience in the schools to see a

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[1] "If the child is to learn directly, with a maximum use of his initiative, it is absolutely essential that the teacher should provide some motive. This implies that the child is to be interested in some fundamental way in the activities in which he is to engage. Instead of thumbing the fundamental facts with his memory in an artificial and effortful manner, 'singsonging' the tables rhythmically, so as to make dull business less dull, the child studies the arithmetic involved in his own life, for the modern teacher brings the two together. The number story, the arithmetical game, playing at adult activities, constructive work, measuring, and other vital interests of the child and community life become increasingly the basis of instruction in number. Such is the pronounced tendency wherever the movement is away from the traditional rote learning or drill." Henry Suzzallo, *The Teaching of Primary Arithmetic;* Houghton Mifflin Company, 1911, p. 63.

"The law of habit is supposed to be that 'practice makes perfect,' or that the nervous system 'grows to the modes in which it is exercised.' But practice without zeal — with equal comfort a success and failure — does not make perfect." Thorndike, *op. cit.* (p. 21), Vol. II, p. 22.

"But so far as we can succeed in focalising our attention exclusively upon the matter boy or girl roused to activity by a new motivating force. I have seen a boy of ten drag along under the ordinary humdrum of school life, deceiving his teacher, his parents, and himself into thinking that he was doing his very best, though the results were pitifully meagre. After a careful study of his idiosyncrasies and the use of motives best suited to his case, I have seen this boy more than double his rate of learning. He passed through the fifth grade in a semester more creditably than he did the fourth grade in a year. It is not at all unlikely that, with proper motivation, with due attention to making the work agreeable, and with other improvements in teaching arithmetic to be noticed later, at least two years may be cut off the present course in arithmetic, and at the same time the work may be done better than it usually is at present.

The work in the schools should be properly motivated, not only because in this way will the school work be done more effectively, but because the ideals developed in school will "carry over" into later life. "Just as the twig is bent, the tree's inclined." If the motives used in the schools are such as will grow naturally into high permanent ideals of life and become the foundation of worthy and permanent purposes, one of the great aims in "preparing for life" will be achieved. The possession of a master purpose is one of man's chief assets. Like the great modern liner, he whose possession it is lays his course over the ocean of life and sails it undeviatingly, paying little heed to the vicissitudes of wind and sea. But not even he sails his course without struggles. The speed of the liner is decreased by adverse winds and great storms; even the greatest of them are obliged at times to turn their prows into the breaking seas and ride them until the storm

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in hand, so far we do make gains in vividness, and the importance for efficient memory processes of concentrated attention is based upon precisely this fact.

"The habit of giving oneself with complete abandon to the undertaking immediately in hand is one of the most significant clues to the securing of an alert and accurate memory." Angell, *op. cit.* (p. 120), p. 197.

abates, when the journey can be resumed. So it is even with the strongest and most purposeful of men. Great sorrows sway them from the courses of their lives; transient but overpowering impulses, their inheritance from a million aimless savage generations, may temporarily control their actions. He whose permanent motives are weak, whose life is not mastered by a great and worthy purpose, may be engulfed never to rise again — a lost hope, a promise unfulfilled. But,

"O well for him whose will is strong!
He suffers, but he will not suffer long."

He turns again upon his course, often atoning for errors by a still higher life until at its end he hears the "well done." With many, such differences find their beginnings in the ideals and motives encouraged in the plastic years of school life. The study of motivation is one of the great problems both for the experimental psychologist and for the practical teacher.

REVIEW QUESTIONS

1. What motivating forces that are effective in the lower grades continue to be effective in the upper grades? (§ 81.)

2. Discuss derived interest and its importance in the upper grades. (§§ 71, 82.)

3. Discuss the rôle of imitation and of ideals as motivating forces. (§ 83.)

4. How may the child's increasing interest in human affairs be made a motivating force for the study of arithmetic? Contrast this interest with a desire to learn what is "useful" to the adult. (§§ 84, 85.)

5. Contrast transient and permanent motives as springs of human conduct. Consider this in relation to the selection of motives. (§§ 72, 86, 87, 88.)

6. Discuss the doctrine that "love of the work" should be made the one sufficient motive. (§ 87.)

7. Contrast the effectiveness of various motives such as necessity imposed by authority and certain "artificial" motives proposed in this book.

8. Contrast the motives to be used in the lower and upper grades, (a) as to the immediateness or remoteness of the motivating elements; (b) as to the general character of the motives whose growth should be encouraged.

PART TWO

SPECIAL PROBLEMS IN THE TEACHING OF ARITHMETIC

CHAPTER V

OBJECTS TO BE ACHIEVED IN ARITHMETIC

89. The purpose of this chapter. — It is not the purpose to consider here the question whether certain topics, such as highest common factor, compound interest, or square root, should be included in the course of study, but rather to point out certain qualities of teaching and certain points of view that are necessary for the best results in elementary school arithmetic and also to indicate certain topics not logically parts of arithmetic which should be included in the course. It matters little whether a certain minor topic is excluded or not, but the general manner of teaching the subject matters greatly. The former can make a difference of only a few weeks of work; the latter affects the whole course.

It is assumed that there is no further need to emphasize the advisability of omitting certain obsolete topics, such as cube root, partial payments, annual interest, unusual and complex fractions, etc. The campaign against them seems to have been effective and they may now be allowed to rest in peace.

90. Importance of clearly defined purposes. — To prevent aimless work and loss of time, it is necessary that the teachers as well as those who make out the course of study have clearly in mind the purposes to be attained in the course as a whole. This includes not only a list of the topics to be taught but also the reasons for the inclusion of each topic and the spirit in which the whole subject is to be treated. The better the subject of arithmetic is taught the more closely are its various parts interrelated, not only in substance, but also in form of treatment. One of the most necessary qualities of a good course in arithmetic is that a few fundamental ideas should be made to stand out clearly and that each of these should be made to serve as many different purposes as possible. Certain ideas whose development is begun in the third and fourth grades should find increasing application in the later years of the course. These ideas cannot be developed properly unless the purposes to which they are to be put later are definitely known to the teachers in the early grades. It is therefore necessary that the teachers in the early grades should know not only the topics of the succeeding grades but also the general method by which they are to be treated; and it is equally necessary that the teachers in the upper grades should know the ideas that were stressed and the general methods employed in the lower grades.

In this chapter, a general outline will be given of the purposes to be achieved. A more detailed account of these purposes, together with a statement of the methods that may be used in achieving them, will be given in the succeeding chapters.

91. Arithmetic proper and allied topics. — The materials to be included in a course in arithmetic in the grammar grades may be divided into two classes. One of these consists of those considerations about numbers and operations on them which are logically necessary parts of the subject of arithmetic. Such topics are addition, subtraction, multiplication, and division of integers, common fractions, and decimals. The other class consists of certain allied topics which are always brought into courses in arithmetic though not logically necessary parts of arithmetic. Such topics are areas of rectangles and triangles, a large amount of information on business usages, and many others.

The real reasons for bringing in such allied topics have not always been clearly understood. There are certain topics about which the child should be informed in his elementary school course but which are not necessary parts of any of the main subjects that constitute his course. The question then arises as to which subjects should be made to include such topics. Thus, the elementary schools should unquestionably afford a bright boy the opportunity to learn something about the real reasons why interest is paid on borrowed money or why commercial discounts are given. Should this opportunity be given in the course in arithmetic or in some other course? The answer to this question is important. If it is decided that such a subject is to be taught in the course in arithmetic, then it should be studied not merely because it affords illustrative material for arithmetic but for the sake of obtaining information about the subject itself. The time given to such topics and the thoroughness with which they are mastered will depend very largely upon the real reason for which they are studied.

The principle which should govern the distribution of such miscellaneous topics among the various standard school subjects is, obviously, that each topic should be placed where it is most naturally allied with the main subject and where its consideration may be affected with the least break in the continuity of that subject.

The line of demarcation between the two classes of topics is not always clear, but it is important to recognize that there is a class of miscellaneous topics of the kind described here and that distribution of these topics among the various school subjects be carefully considered. If this is not done, it is very probable that unfortunate gaps will be left in the child's education, gaps which even elaborate later education may fail to fill. This is particularly true during a period, such as we have been passing through during the last generation, in which the general tendency has been to omit topics formerly included in the various school subjects. Recently the writer in conversation with an educated woman, a graduate of one of the well-known American universities and of various schools that precede the university, a teacher in a strong high school, found that she believed Jupiter to be larger than the Sun (she had been reading about the recent measurement of the diameter of Betelgeuze), that she had no idea of the difference between a planet and a fixed star or of the solar system as distinct from other heavenly bodies. In the days of the writer's elementary school life information about this matter was included in the course in geography and was not slurred over. The simple truth is that this young lady's school course was such as to *prevent* her from obtaining the information which would have saved her from exposing an ignorance that was astounding. The duty of our schools is not only to furnish information which is *necessary* for the most stupid but to afford the opportunity for the more favored to acquire information which is their birthright and the culture with which it should be their privilege to adorn their lives.

92. Topics necessarily included as logical parts of arithmetic. — Logically arithmetic consists of the study of numbers and of the fundamental operations on them. The nature of integers and of fractions, common and decimal, and the operations of addition, subtraction, multiplication, and division with these numbers constitute pure arithmetic.[1] The other topics found in our courses in arithmetic are illustrative, practical applications, or miscellaneous topics introduced because they cause less disturbance in arithmetic than they would in some other subject.

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[1] "For the ordinary purposes of nontechnical daily life we need little of pure arithmetic beyond (1) counting, the knowledge of numbers and their representation to billions (the English thousand millions), (2) addition and multiplication of integers, of decimal fractions with not more than three decimal places, and of simple common fractions, (3) subtraction of integers and decimal fractions, and (4) a little of division. Of applied arithmetic we need (1) a few tables of denominate numbers, (2) the simpler problems in reduction of such numbers, as from pounds to ounces, (3) a slight amount concerning addition and multiplication of such numbers, (4) some simple numerical geometry, including the mensuration of rectangles and parallelepipeds, and (5) enough of percentage to compute a commercial discount and the simple interest on a note." Smith, *op. cit.* (p. 111), p. 21.

In this part of the course it is necessary that certain number facts be memorized or rather habituated so as to be ready for instant use with the expenditure of very little energy in recalling them. Thus "7 × 8" should cause "56" to arise instantly in the mind without the slightest feeling of effort. The number facts to be thus memorized are the addition and multiplication tables, together with certain other facts immediately connected with them which will be noted later (see sections 105, 109, 110, 113). The methods of teaching these combinations are considered in Chapter VI.

There are certain well-defined combinations of steps, constituting the fundamental operations, which should be so habituated as to be taken in their proper order without the slightest necessity for reflection on the nature of the steps or the reasons for their order. All the fundamental processes with the various kinds of numbers should be habituated in this manner. For example, each step in long division and the order in which the steps are taken must become a matter of pure habit. It must be clearly understood that the method of teaching the processes and the possible effort to understand them when they are *first introduced* is a very different matter from the question which we are now considering; namely, the ultimate result to be attained. The methods of teaching these operations are considered in Chapter VII.

93. Standard of accuracy — checking. — The aim in arithmetic should be to make all results absolutely correct. All operations and methods should be checked in such a manner as to create a very high degree of probability that they are right. No work should be regarded as satisfactory which is not actually 100 per cent correct. In a bank, work that is 95 per cent or even 99 per cent correct is a failure.[1] In an engineer's office one third and often one half of the time used for computing is consumed in checking. The results must be exactly accurate; otherwise death or destruction may result. Methods of checking should be studied with the same degree of care and thoroughness as the operations themselves. It should be understood that the time required for checking an operation may be as great as that required to perform the operation. In this book, methods of checking are considered under the various operations; at this point the purpose is simply to consider the general attitude toward checking and accuracy. If the program suggested above is adopted, methods for carrying it out may be found readily and put into operation. Too frequently little or no attention has been paid to the checking up of the work; teachers have been satisfied with 90 per cent or 95 per cent work even in the upper grades. In practical life only 100 per cent accuracy in arithmetic will do and this should be the case in the schools.[1] There is much evidence to show that this is now far from being the case and that great improvement is absolutely necessary if the schools are to meet the standards of practical life.[2]

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[1] "Real life demands a rating of 100 per cent in nearly all calculations, but it generally gives the calculator all the time he requires. The teacher must, therefore, secure this to

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the greatest extent possible, rapidity being exacted only in so far as it tends to make work more accurate by calling for greater attention." J. H. Walsh, *Methods of Arithmetic*, 1911, D. C. Heath & Co., p. 30.

"It appears, at least to the author, imperative that checking should be taught and required until a pupil can add single columns of ten digits with not over one error in twenty columns." Thorndike, *op. cit.* (p. 42), p. 33.

[1] "Checks are of many kinds. The teacher may serve as a valuable check upon the work of the pupil, but many teachers, by habitually accepting inaccurate and untidy work, do not stimulate the pupil to greater accuracy." Brown and Coffman, *op. cit.* (p. 31), p. 57.

"If one wishes to produce number habits with the least expenditure of energy and time, he should permit no exception — the correct answer must be given every time. One never gets right results by praising wrong answers." Brown and Coffman, *ibid.*, p. 103.

"One of the best ways of making pupils think is to have them verify their results. This is, again, the value of Spencer's 'natural consequences' idea. . . . The check for arithmetic problems makes the pupils go over the work again with care. Let practice teachers try out their principles upon live pupils and they begin for the first time to understand the problems and methods of teaching. Wild guessing by an erratic pupil is steadied by requiring him to carry out his guesses to verify them." Charters, *op. cit.* (p. 123), p. 201.

[2] "The evidence of inaccurate computation in our schools is overwhelming. It is not unusual to find a group of seniors in high school not twenty-five per cent of whom can add

94. Addition relations, subtraction relations, etc. — It is trite to remark that skill in arithmetic consists in the ability to perform the fundamental operations rapidly and accurately, together with ability to decide how these operations must be combined to solve practical or concrete problems. The first step in deciding how to combine the fundamental operations in the solution of problems is to decide which simple relations require addition, which require subtraction, etc. John has 15 marbles and buys 12 more; he will then have how many? This should at once be recognized as an addition relation. Again, John has 15 marbles and James has 27; how many more marbles has James than John? This should, of course, be recognized at once as a subtraction relation.

The point that is being made here is that the various situations that give rise to addition should be studied and the idea of *adding* should be connected with them. Similarly subtraction relations should be studied and the idea of *subtracting* connected with them, and so on for multiplication and division. This matter is discussed more fully in sections 183–187. A clear understanding on the part of the teacher of the obvious fact that all there is to be done in a course in arithmetic consists of memorizing the fundamental number relations, habituating the fundamental operations, and acquiring skill in applying these to the solution of problems will do much to define the objects to be achieved in arithmetic.

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ten numbers of four digits each and secure the correct result the first time." Brown and Coffman, *op. cit.* (p. 31), p. 52.

"Arithmetic as a tool is almost useless unless it has an edge keen enough to do its work with considerable speed and absolute accuracy. . . . Teachers permit and sometimes encourage inaccuracy by giving high grades for the correct process, even if the result is absurd. Pupils soon form the habit of accuracy when they find that inaccurate results are always marked zero." N. E. A. *Proceedings*, 1906, p. 101.

"Some of the pupils of our schools seem to believe that accuracy in computation is of so little moment as to be almost beneath their intellectual dignity. It is hard to eradicate such an idea after the pupil reaches the high school age." Brown and Coffman, *ibid.*, p. 53.

95. Weights and measures — geometrical relations. — Certain tables of weights and measures should be memorized so as to be ready for instant and effortless recall. Among these should be included all the simpler units of measure and their relations, such as inches, feet, yards; ounces, pounds; cents, dollars, etc. It is not the purpose here to discuss the question as to which tables should be included and which should be omitted but to indicate the character of the work to be done on those which it is decided to include. Absolute readiness and spontaneity of recall is the desideratum.

Early in the grades the child learns to find the area of a rectangle and the volume of a rectangular solid, and later in the course considerable information is obtained about other geometrical relations. Weights and measures, areas and volumes are instances of topics not logically necessary in arithmetic which have been placed in this course because they belong more naturally there than elsewhere.

96. Topics from physics. — Topics, such as speeds and distances, temperature and the thermometer, are included in nearly all courses in arithmetic. Every child in the second grade knows that if a person walks three miles in one hour and keeps this up for two hours he will go a total distance of six miles. The mature person generalizes this fact into the statement:

$$\text{Speed} \times \text{time} = \text{distance}.$$

In this form it is entirely unintelligible to the untutored child, but this general fact, formulated for effective use, as is done here, is of such fundamental importance that the necessary steps to develop it should be taken. This can be done as early as the fourth or fifth grade.

Even such a subject as specific weights may be of the highest interest to a child in the grades, and no doubt the class in arithmetic is the most natural place where it should be considered.

Recently I heard a boy of eight inquire with a great deal of interest as to which kind of wood, light or heavy, would float the better. His opinion was that heavy wood was stronger and, therefore, would "carry more" in water. This child at the normal age of children in the third grade was prepared for an intelligent and interesting consideration of some of the questions connected with specific weights. In fact, he told with keen interest that he knew why a concrete ship would float: namely, because there was enough air inside it to make the whole thing lighter than water. This, he said, had been told him by his mother, and it made a sufficient impression on him so that he remembered the statement very definitely and clearly a week after it had been made. The truth probably is that children have an effective interest in many questions related to their physical environment, the consideration of which we are now in the habit of postponing until a course in physics is reached somewhere in high school. One of our stale notions has been, and still is, that nothing should be considered in connection with a subject until the child is ready to study a more or less comprehensive and logical treatment, such as is developed in a textbook used somewhere in the school system. We too often permit the period of intense curiosity and interest to pass by without making proper use of it. When the facts have become commonplace, the interest in their explanations has subsided and is reawakened with difficulty, if at all. It is important, therefore, that a sharp watch be kept on all the possible interests which the child may develop and to seek as far as is at all practicable to furnish the information and explanation appropriate to such interest.[1]

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[1] William James goes further in this direction than the majority of educational psychologists. Nevertheless, the following passage probably contains a very large element of truth:

"If a boy grows up alone at the age of games and sports and learns neither to play ball, nor row, nor sail, nor ride, nor skate, nor fish, nor shoot, probably he will be sedentary to the end of his days; and, though the best of opportunities be afforded him for learning these

97. Information about affairs of business. — When we reflect upon the purposes of a school curriculum, it is, of course, apparent that we want, not only to develop the ability to compute interest, for example, but that we also want to impart clear and comprehensive information about what interest is, how it comes to be, etc. Indeed, it may be remarked in passing that in solving problems in interest, there is no new arithmetic whatsoever involved; the only new element is the social situation which gives rise to the problem — but more about that in another place. The point here is that it is a part of the function of the school curriculum to provide information about the nature and origin of interest, about the fact that interest is being paid all about us, and about the great social significance which it has come to have. If an adult person does not know these things, he should be regarded as not having a fair education, and it is the business of the common schools to give people a fair education. It is probably true that the class in arithmetic is the most natural place to do this work. There is, of course, another reason why it should be done here: namely, that when the nature of interest is understood, the difficulties of the problems have practically all vanished. Remarks similar to these apply equally to all applications of percentage. It may be said about all of these that the social phenomena which underlie the problems should be the point of attack. This is true of such subjects as stocks and bonds,

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things later, it is a hundred to one that he will pass them by and shrink back from the effort of taking those necessary first steps the prospect of which, at an earlier age, would have filled him with eager delight.

"In all pedagogy the great thing is to strike the iron while hot, and to seize the wave of the pupils' interest in each successive subject before its ebb has come, so that knowledge may be got and a habit of skill acquired — a headway of interest, in short, secured, on which afterward the individual may float. There is a happy moment for fixing skill in drawing, for making boys collectors in natural history, and presently dissectors and botanists; then for initiating them into the harmonies of mechanics and the wonders of physical and chemical law. Later, introspective psychology and the metaphysical and religious mysteries take their turn; and, last of all, the drama of human affairs and worldly wisdom in the widest sense of the term." *Op. cit.* (p. 57), p. 404.

insurance, taxes, investing of money, etc. The actual method of handling these topics will be considered in detail in Chapters XIII–XXI.[1]

98. The use of language. — Our arithmetics contain a great many so-called story problems. One of the results to be achieved in arithmetic is ability to read such problems intelligently and to understand the conditions which are stated in them. This is an achievement of no small importance and should receive separate and independent attention. The ability to make accurate statements is, of course, more difficult to acquire than the ability to read the printed page. There are few, if any, parts of the school curriculum in which a child can be so effectively trained to make clear-cut and accurate statements as in arithmetic. In the teaching of English composition, it is too frequently true that the subject matter of the composition is sufficiently vague or elastic to permit the use of several different expressions of somewhat similar meanings, without making it possible for the teacher to determine whether or not exactly the proper expression has been used. It would seem that the sequence of events in English composition should be:

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[1] "Everywhere in these days the American teacher and the educational writer speak of the social aims of educations. More than ever before, the social consciousness of the teacher enlarges. . . .

"In spite of an increasing movement toward specific vocational training — as seen in the industrial education movement, for example — there has been a reactionary defense of the elementary school as an institution for every general training in the things that are socially fundamental and common." Suzzallo, *op. cit.* (p. 165), p. 17.

"Arithmetic is not a subject in which only the skills of calculation are cultivated; it is one that contributes social insight, just as history and geography do." Suzzallo, *ibid.*, p. 18.

"We are beginning to appreciate the extent to which arithmetic may be made to contribute to social insight. Pupils are shown how the world uses its mathematics by visits to centers of commercial and industrial activity, and the setting of the problem and its solutions are studied. Instead of devoting all of the time to the solution of problems about a factory or a bank, the modern class occasionally visits these institutions. A good course in arithmetic to-day includes a consideration of the following topics, all of which have been introduced recently: the saving and loaning of money; the investing of money; modern banking methods; keeping of simple accounts; a study of tax levies; public expenditures and insurance from the social point of view. Much of the work in these topics must be of the informational kind and belongs quite as much to a course in civics and economics as to arithmetic." Brown and Coffman, *op. cit.* (p. 31), p. 250.

a. A definite subject, about which it is desired to make very definite and clear-cut statements.

b. The expression in conventional form of precisely these statements and not of some other statements more or less similar in their meanings.

It is too often true that, in writing or speaking, a convenient phrase or expression is retained, even though it does not express exactly the idea that it is intended to convey. The mind is switched over, possibly even deceiving itself, so that in the end it is believed that the statement actually used is the one which it was originally intended to make. Changing one's attitude of mind to accord with a convenient expression is an all too common occurrence, and it is not confined to the young only. When talking or writing about arithmetic this cannot be done. The subject matter is so clearly defined, its parts so definitely and logically connected that it becomes absolutely necessary to suit the words to the facts which are to be represented. The wobbling of the thought to accord with a convenient and happy-sounding sentence becomes impossible, and the real effort which must always be made when language is to be used effectively and accurately—namely, to make the language express precisely what is in the mind—must perforce be made. In so far as the use of English in the work of arithmetic is concerned, the teacher of arithmetic should be equally responsible with the teacher of English for the child's correct and effective use of language, and the teacher of arithmetic should not forget that in some respects she is in a position to meet this responsibility more effectively than is the teacher of English.

It should also be kept in mind that correct and definite use of language reacts upon the thought in the way of making it definite and correct. It is a grave question how far intellectual processes could have developed in man if he had not acquired the use of articulate and ideated speech, and it is an equally grave question how far a person can now proceed in clear and effective thinking without putting his thought into clear and effective language expression. We are, of course, acquainted with the person who says, "I know that perfectly, but I can't say it." That may be true in some cases, but in most cases when an attempt to "say it" is really made, it will be found that the idea is not clear, and presently we shall hear the remark, "Well, I'm not exactly certain about it." Without the language expression, the idea seems to remain vague and undifferentiated — a haze through which the landscape of thought appears vague and uncertain.[1]

99. Understanding principles versus remembering rules. — Perhaps the most important point established by the experimental investigators in the problem of formal discipline is that concepts and principles carry over to other mental activities much more completely than do skills acquired *merely* by empirical practice. Hence, it is of fundamental importance that effort be made to understand principles rather than to memorize rules whose development has not been mastered.[2]

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[1] "A common criticism of the schools of to-day is that the pupils have been permitted to become lax and careless in thought and in expression. The modern pupil is expected to study many things which were not taught in the schools of the last generation, but there is truth in the assertion that no small part of our knowledge is superficial and inaccurate, 'a collection of vague ideas rather than clear-cut notions about definite things.' It is asserted that while the pupils of to-day can think and write on more subjects than could the students of former years, their expressions are less clear and coherent." Brown and Coffman, *op. cit.*, (p. 31), p. 43.

"The expression used by the pupil often exposes the mental process; teachers can improve accuracy of statement by frequently insisting upon full and concise explanation." Brown and Coffman, *ibid.*, p. 44.

[2] "More emphasis is being placed to-day on the mastery of principles and less on the mastery of rules and definitions, than was the case in former years. The pupil who fails to grasp the few simple underlying principles of the subject has seen but little of the beauty and the simplicity of arithmetic. The number of distinct mathematical principles involved is surprisingly small; but unless a pupil grasps these principles, the whole subject lacks organic unity in his mind. It is important to analyze and to classify topics; to be able to distinguish points of similarity and of difference; to develop the power to see relations between topics that are based on the same underlying principles. As McMurry says,

A modified version of Mark Twain's story of his ascent of the Alps will illustrate the point: Mark Twain made the most elaborate preparation for ascending the Alps that had yet been made. From the point of view of "practical" educators, his preparation was ideal and complete. He read every book that he could find on Alpine climbing, talked with all people to whom he could get access who had ever climbed a mountain. Then he procured a very large book of blank paper. At the head of each page he entered the name of one of the obstacles which may be encountered in Alpine ascent; and under the name of this obstacle, he entered the most approved ways of overcoming it. In this way he was prepared to meet every conceivable difficulty with the combined wisdom and skill of all those who had ever encountered them. Surely, he had made a complete preparation for the journey. Attired in a dress suit and mounted on a donkey, he headed his expedition as it started from in front of a large and fashionable hotel on a fine summer afternoon. The party had not gone far before "a precipice ascending" was encountered. "A precipice ascending" — Mark Twain was prepared for the emergency. "A precipice ascending should be ascended by the use of ladders," etc. That is what his book said. It also said that at the first precipice the cattle and mules should be left behind. When the party had reached the top of the precipice, they found themselves confronted by a "precipice descending." The book explained how to descend precipices, and the feat was accomplished. However, some cans of nitroglycerin had been left at

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'Merely to go through a textbook without picking up the strings and tying them together is to fail in a most essential thing.' . . . Spencer says, 'When a man's knowledge is not in order, the more of it he has the greater will be his confusion of thought. When the facts are not organized into faculty, the greater the mass of them the more will the mind struggle under its burden, hampered instead of helped by its acquisition.' . . .

"The custom has been to group problems more or less into series involving the same processes but different in all other respects. The present tendency is to 'attain a more approximate unity within the subject matter of the problems themselves.'" Brown and Coffman, *op. cit.* (p. 31), p. 360.

the foot of the first precipice, and one of the mules, feeling his ancestral urge, began to kick. When the smoke had cleared away, the mule was gone; and instead of two precipices, one ascending and one descending, there was a large chasm. Again the "book of wisdom" was consulted, and it was found that a chasm should be bridged and the animals taken across. So the chasm was crossed in the approved manner. However, one of the cows refused to cross the bridge, and it was decided to leave the obstreperous beast behind and deprive her of a share in the glory of scaling the Alps. They had gone but three or four hundred yards when the cow walked around the chasm, joined the caravan, and all proceeded merrily. It is to be feared that much of our so-called "practical" education is of the type that Mark Twain is satirizing in this story.

We hear much these days about the practical *versus* the impractical. It is urged that only those things be brought into the school which will be of direct and practical value to man. This all seems simple enough, but in reality it is very far from simple; the problem is exceedingly difficult. What is practical and what is not? That is not an easy question to answer. One child is taught, for instance, to compute interest, according to a certain method which is in general use. Another child is not taught any particular method, but somehow the child gets a thoroughgoing grasp of the general principle of interest, and he also possibly gets into the habit of getting things done. It may very well be that the latter child has the more practical education.

The fundamental fact is that every day is a new day. The old adage that "there is nothing new under the sun" has an accompanying one which is equally true, and that is that "everything under the sun is new." The fact which must be everlastingly borne in mind is that we must be prepared to adjust ourselves to changes of conditions. Ability to reflect on a situation and to adjust oneself to it, even though it may be done somewhat slowly, may be of vastly greater practical importance than immediate and great facility in performing all the practical arithmetic tricks that have ever been used.

Hence, persistent effort should be made to understand the principles involved rather than merely to remember rules.[1] This applies to all topics in arithmetic except the combinations that are to be memorized and the operations that are to be habituated (see section 92). This general principle will be referred to as the teaching of each topic is considered.

100. Understanding the human significance of the subjects studied. — Contrary to what frequently seems to be assumed, the child may be led more easily to take an absolute delight and interest in performing simple routine operations which require little or no thought-effort than in solving the most vital and, to the intelligent adult, most interesting concrete problems.[2] By the use of such means of motivation as have been discussed elsewhere, it is possible to raise this interest to a high pitch and thus to secure the maximum learning efficiency when the fundamental processes are being studied and habituated. In a very vital sense, however, it is important that the child should learn to see the human significance of the work that he is doing. He should be led to feel that the arithmetic which he is studying is an element of that rushing stream of adult life which at first he views as one detached but which is gradually to engulf him and make him a part of itself. As the course in arithmetic progresses, he should be made to understand increasingly that arithmetic is one of the means that the race of men is using to perform the tasks which their lives demand of them. When studying the subject of interest, for example, the child should be made to understand in a first-hand way how widespread and important are the borrowing of money and the consequent paying of interest. He should learn that borrowing money is by no means a sign of economic inferiority on the part of the borrower, that business concerns of the very highest financial responsibility borrow large sums from time to time from the banks, paying interest for the same. He should know that practically all active business concerns are borrowers of money at one time or another. He should understand that a farmer who borrows money from the bank for a few months in a year in order to put in more and better crops or to provide equipment for his business in any way whatsoever may be a much more successful farmer financially than his neighbor who never borrows a cent. In short, the child should learn to understand as much as may naturally be brought within his comprehension of the economic and social significance of the process of borrowing money and of the custom of paying interest for money which is borrowed.

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[1] "The worst thing about empiricism in every department of human activity is that it leads to a blind observance of rule and routine. The mark of the empiric is that he is helpless in the face of new circumstances; the mark of the scientific worker is that he has power in grappling with the new and the untried: he is master of principles which he can effectively apply under novel conditions. The one is a slave of the past; the other is a director of the future." McLellan and Dewey, *op. cit.* (p. 31), p. 9.

See note from James on pages 56, 57.

[2] "Further, computation is not dull if the pupil can compute. He does not himself object to its barrenness of vital meaning, so long as the barrenness of failure is prevented." Thorndike, *op. cit.* (p. 42), p. 110.

It is not contended that an understanding on the part of children, especially in the lower grades, of the significance of any process to adults will tend to motivate it. What is contended is that an understanding of the human significance of the process is a part of the educational aims to be achieved, quite independently of whether or not it motivates the study of arithmetic. Recently the writer heard an official in a large school system explain elaborate "projects" by means of which it was proposed to motivate the study of arithmetic. Building chicken coops, raising Belgian hares, etc., were to furnish the bases for extended chains of examples. An old superintendent inquired, "What is the purpose of these projects?" "To motivate the study of arithmetic," was the answer. The superintendent replied, "It seems well enough, but it will not work. When your children get through with their projects and their chains of problems, they will hate the work. By using games of rivalry, I can get them to like arithmetic so they will leave every other study in the curriculum for it." In an important sense, this superintendent was right. The study of the human significance of the process instead of being a motivating element may require special motivating devices of its own. But even so, it is important and should not be neglected. In the upper grades, however, when the child begins to identify himself more and more, though only gradually and in part, with the adult community, and when the desire to take his part in the grown-up world increases in strength, this understanding of the human significance of the work he is doing is likely to help to create and sustain — chiefly sustain — his interest.

101. Development of the critical attitude. — It is one of the fundamental facts of mental processes that incorrect conclusions are often drawn at first. This is true of the simple operations of arithmetic, such as addition and multiplication, and it is likewise true of a course of reasoning by means of which the method for solving a problem is discovered. We have already noted the importance of checking fundamental operations (section 93). It is equally important that the methods of solving problems should be studied carefully. The mind cannot attend to everything at one sweep, and it happens frequently that a subject appears perfectly clear because some phases of it are left entirely unattended to, and that a course of reasoning is at first regarded as valid which on second thought, when the elements neglected are considered, appears obviously wrong. The habit of going over a course of reasoning or argumentation again and again, with a well-formed suspicion that there probably is something wrong somewhere and with an endeavor to find such shortcomings, if they exist, may be regarded as the development of the critical attitude. The difference between those who reason well and those who do not lies not so much in that one is more clear-headed than another, but in the fact that one has formed the habit of carefully scrutinizing each part of a course of reasoning with the attitude of mind that something is probably wrong, thus making a well-defined endeavor to test to a certainty each part of the reasoning; while the other seeks to take in the whole situation at a glance, making possibly a very strenuous effort but making no careful and studied scrutiny of the various parts of the argument.

REVIEW QUESTIONS

1. Discuss the statement that the teacher in the elementary grades should know, not only the topics of the succeeding grades, but also the general method by which they are to be treated. (§ 90.)

2. Name the most important topics which are usually taught in a course in arithmetic but which are not logically necessary for the course. What reasons are there for including such topics? (§§ 91, 92.)

3. Name the most important topics that are logically necessary parts of a course in arithmetic. (§ 92.)

4. What is the standard of accuracy in the practical applications of arithmetic? How can such a standard be maintained in the schools? (§ 93.)

5. What tables of denominate numbers should be included in the course in arithmetic? How well should these tables be memorized? (§ 95.)

6. What topics from physics should be included in the course in arithmetic? Why should not these topics be postponed until the subject of physics is studied in the high school? (§ 96.)

7. Discuss the reasons for introducing information about business usages in arithmetic. (§ 97.)

8. What is the chief difficulty in making practical application of rules when the principles underlying them are not understood? Is it possible to predict situations so closely as to formulate rules that will meet them without alteration or readjustment? (§ 99.)

9. Discuss the reasons for teaching children something about the human significance of arithmetic. Consider the extent to which this should be carried in the lower and in the upper grades respectively. (§ 100.)

10. Distinguish between critical attitude and critical skill. Discuss the value of critical attitude. If one of the purposes of the course of arithmetic is to develop a critical attitude, how will this influence the general character of the course? (§ 101.)

CHAPTER VI

NUMBERS AND NUMBER COMBINATIONS

102. The number concept. — When the child first hears numbers used, he thinks they are the names of things. Mother holds up one finger to a child of three and says, "One finger." The child holds up a finger and repeats, "One finger." The mother then holds up two fingers and says, "Two fingers." The child does likewise. Then she says, "Show us one finger," and the child holds up the first finger and says, "One finger." "Show us two fingers," says the mother, and the child holds up one finger, the second one of those first held up, and says, "Two fingers." However, by the age of five, or six at the latest, the child has developed the notion that numbers tell "how many," and about this time too the child begins to talk about numbers without reference to any particular thing — that is, about numbers in the abstract. The notion of numbers no doubt arises from the process of counting, that is, of taking 1 and 1 to get 2, 2 and 1 to get 3, etc.[1] The process of abstraction is, of course, the result of many countings of many different groups of things.

Any distinct objects may serve as a basis for counting. Thus, we may have one dog, three cats, eight sheep, or we may have twelve animals. The counting proceeds entirely independently of any notion of equality among the objects counted. All that is necessary is that they may readily be regarded as separate things.

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[1] "The most fundamentally important thing for beginners is to learn to count. Counting consists in thinking objects serially related and, later, in thinking the symbols that represent those relationships. Consequently, it is not synonymous with measurement, for that consists of comparing or measuring objects alike in nature — as blocks, two lines, two sides of a room, some shorter, some longer." Brown and Coffman, *op. cit.* (p. 31), p. 144.

The process of measurement in the usual sense of the word is much more difficult and comes later, after the number concept has been developed by counting. It is believed that the effort to get the child to develop his number concept originally by measuring is wholly unnatural and very mischievous. If the word "measuring" is used in its usual sense, it can easily be shown that measuring is a more complicated process than counting, since it includes counting plus another process which is itself difficult. To illustrate: A child is to develop the idea of six by measuring a line six feet long. This involves the repeated application of the foot rule to successive parts of the line and care on the part of the child that each application begins just where the preceding one ended — a process which is by no means easy and which absorbs his full attention. Besides this, he is required to count the number of applications of the rule. It would be a miracle if this complicated process should lead more naturally and easily to the concept "six" than the process of counting six objects of one kind, then six of another kind, etc. Our race learned numbers by counting and not by measuring; numbers were in the world long before man learned to measure.

The abstruse theory of numbers, according to which all numbers represent the results of measure, causes more trouble than it is worth. Counting may, of course, be regarded as measuring a group of objects by using one object as a unit, if one chooses to use language that way; but little if anything is gained by it, and much is lost in that considerable effort is required of the reader in keeping in mind such unusual meanings of words that are in common use.[1]

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[1] McLellan and Dewey are perhaps the most prominent among the writers who have adopted the point of view that numbers arise from measurement. At the same time they assert most unmistakably that counting is measuring.

"It has been said that number originates from measurement; that it is a statement of the numerical value of something. But we are accustomed to distinguish counting, *i.e.* numeration, numbering, from measuring. It is usually said that we count objects, particular things

Finally, it has been proposed seriously that all numbers be regarded as ratios even from the very beginning. This proposal need not be regarded seriously since it involves procedures so unnatural and cumbersome that none but the blindly enthusiastic will seek to put it into practice. The idea of ratio is valuable, but its development belongs to a later stage.

103. Analysis of numbers and numbers for use. — There has been and still is considerable divergence in the general purpose of early number work. Some believe in a certain type of thoroughness which consists in exhausting all the properties of one number before going on to the next. According to this view, before proceeding to the number 7, the child should have complete knowledge about all numbers up to and including 6. This involves not only the addition combinations whose sums are 6 or less but also all subtraction, multiplication, and division combinations that are within this limit, together with applications to a large variety of concrete and practical uses. Grube advocated limiting the first year's work entirely to numbers up to 10. As quoted by Smith,[1] Grube says: "In the thorough way in which I wish arithmetic taught, one year is not too long for this important part of the work. In regard to extent the pupil has not, apparently, gained very much; he knows only the numbers 1 to 10 — but he knows them."

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or qualities, to see how many of them there are, while we measure a particular object or quality to see how much of it there is. We count chairs, beds, splints, feet, eyes, children, stamens, etc. simply to get their sum total, the *how many;* we measure distance, weight, bulk, price, cost, etc. to see *how much* there is. Some writers say that these 'two kinds' of quantity, which they call quantity of magnitude (how much) and quantity of multitude (how many) are entirely distinct. Nevertheless, all counting is measuring, and all measuring is counting. When we count up the number of particular books in a library, we *measure* the library — find out how much it amounts to as a library; when we count the days of the year, we measure the time value of the year; when we count the children in a class, we measure the class as a whole — it is a large or a small class, etc. When we count stamens or pistils, we measure the flower. In short, when we *count*, we *measure*." *Op. cit.* (p. 31), pp. 47, 48.

[1] *Op. cit.* (p. 111), p. 90.

This extreme practice never gained firm foothold in America and of late years the tendency has been further away from it. Twenty-five years ago, McLellan and Dewey stated (not with approval) a doctrine from which we have been departing rather rapidly: "Of this we have had example in the prevailing method of primary number teaching. The child is from the beginning drilled in the 'analysis' of numbers till he knows or is supposed to know 'all that can be done with numbers.' It appears to be forgotten that he may and should perform many operations and reach definite results by implicitly using the ideas they involve long before these ideas can be explicitly developed in consciousness."[1]

Only those who do not know what it means "to know all that can be done with numbers" and what a "complete analysis" implies would ever advocate such a program for children. They are not capable of developing interest in detailed "analysis." Children are doers of things and not philosophers. The only way that numbers can be "used" is by means of the four fundamental operations. For this reason it has now come to be the general practice to develop numbers, say as far as twenty, rather rapidly and to learn at least some of the fundamental combinations as early as possible. The order in which these operations are learned is considered in section 116. There is a distinct tendency not to waste too much time on the "nature" of the operations. Time and maturity will deepen the understanding.

104. Concrete basis for number combinations. — While the fundamental number combinations must finally be memorized

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[1] "If facts [of the fundamental operations] are presented in their proper connection as stimulating and directing the primary mental activities, the child is slowly but surely feeling his way towards a conscious recognition of the nature of the process. This unconscious growth towards a reflective grasp of number relations is seriously retarded by untimely analysis — untimely, because it appeals to a power of reflection which is as yet undeveloped." McLellan and Dewey, *op. cit.* (p. 31), p. 33.

as number combinations so that they may be recalled immediately wherever needed without any thought whatever of concrete things, they must be derived in the first place from the child's experience with objects. Thus, 3 + 4 = 7 must be learned in the first place by placing a group of 4 objects with a group of 3 objects and counting the resulting group. The child should learn this fact in such manner that in case of doubt he would know how to obtain the result again from fundamental sources. This study of the concrete will not only furnish a method for finding the sum in case it should be forgotten, but it will develop, in an organic way, a correct idea of the nature of addition. The study of 3 × 4 = 12 should be made upon precisely the same basis. Thus, not only the fact that 3 × 4 = 12 will be made clear but also the nature of multiplication.

It may not be necessary to study all the forty-five addition combinations this way. When a child has experienced in detail the concrete meaning of 3 + 4 = 7 and of other similar combinations, he is ready to infer that other addition combinations have a similar significance. Of course, no sane teacher would illustrate by objects the meaning of such an expression as 394 + 618. The only practical question left for decision is the exact point where the concrete basis of the operations shall be omitted. Probably no rule can be made for all classes and all individuals. The general principle may be laid down, however, that as soon as the child can find the value of a new combination without recourse to the counting of things, just so soon should he be permitted to do so. The general idea of a *possible* concrete basis will carry over from the combinations which have been studied in the concrete to the other combinations. The child will know that a concrete content may be given to all the combinations if one chooses to do so.

The reason for the desirability of studying combinations in the concrete may be expressed in terms of a more general principle: *Everything that is learned should be related to previous experience.* The new should be apprehended not as entirely new but as a modification and extension of the old. Compare sections 63, 64. The old story about Antæus who was invincible so long as his feet touched the ground still has its significance. To throw him, Hercules had to lift him off the earth. For one thing let us keep our feet on the ground.

On the other hand, the concrete work should not be carried too far. Classes have dawdled over a few combinations for a year, repeating *ad nauseam* concrete illustrations with which the child is as familiar as the teacher — illustrations which are possibly more tedious and uninteresting to the child than they would be to an adult.[1]

Concrete work has sometimes been criticized because the material used is not such as will occur in practical life. This criticism would seem to have no valid basis. The initial concrete work is not expected to carry over directly into practical life. Its purpose is to develop the idea of numbers and the number combinations; it is these that carry over into other school work and into practical life.[2]

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[1] "Objective teaching is fundamental but purely preparatory. The child ought to pass from objects and sense-impressions, through images of various degrees of abbreviation, to symbols and the abstract concepts for which they stand. But in American practice a sharp jump is usually made from concrete objects to abstract symbols. The transition through adequate transitional imagery is not made." Suzzallo, *op. cit.* (p. 165), p. 74.

"The great desideratum in the first year's work is facility in handling *numbers*, not in solving applied problems. 'Tell me a story about four,' is harmless enough at first, although there is no 'story' told; but it gets to be a very old story before the year is done. Children like rapid work in pure number; one has but to step into a class whose teacher is awake to this idea, to realize the fact; and to dawdle through the year with nothing but 'story' telling about number not only leaves ungratified a natural desire, but it sows the seed of poor number work thereafter." Smith, *op. cit.* (p. 111), p. 115.

[2] "Primary children count, add, etc. with things they will never be concerned with in life. Lentils, sticks, tablets, and the like are the stock objective stuff of the schools, and to a considerable degree this will always be the case.

"The playing at store, the use of actual applications of the tables of weights and measures are cases that might be cited." Suzzallo, *ibid.*, p. 48.

105. Number combinations to be made automatic. — While facts, such as $3 + 4 = 7$ and $3 \times 4 = 12$, should be learned through experience with the concrete, the final aim should be to memorize them so perfectly that they will be reproduced spontaneously and without effort or reflection whenever they are needed. They should be so memorized, not for a season or in the presence of a peculiar environment, but so that they may be reproduced automatically any time during life and under any conditions. It is, of course, well known that facts may be memorized in such a way as to enable one to recall them under certain conditions and for a certain time. The writer can easily remember the names and the faces of the members of one of his classes *provided* he sees them in the classroom. It is different when meeting them socially. Somebody has said that one would not know his own mother if he were to meet her in China or in an African wilderness. During the last ten years, the writer has always had one or two telephones, and at no time has he even hesitated to call either one correctly. At the present time, however, he is utterly unable to recall any one of these numbers except those which he is now using.

The addition and multiplication combinations are to be memorized "for keeps." That requires constant reviews, constant use in association with other things, and both of these over a long period of time.

106. Comparative difficulty of addition combinations. — There are certain addition and multiplication combinations which are generally more difficult than others. Investigations on this point have been made in several quarters, and the results seem to be sufficiently in agreement to make them significant:

Holloway[1] made a study of the difficulty of the addition combinations for children in the first and second grades and of multiplication combinations for children in the third grade. The following table shows his results in so far as they bear on our present topic.

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[1] Harry Vance Holloway, *An Experimental Study to Determine the Relative Difficulty of the Elementary Combinations in Addition and Multiplication*, 1914.

The numbers of errors made by 1065 first and second grade children in Holloway's investigation were:

9+8 (95)
9+7 (90)
9+6 (82)
8+7 (69)
8+5 (68)
8+6 (66)
7+5 (56)
9+4 (51)
7+6 (50)
9+5 (49)
7+4 (48)
9+3 (43)
8+3 (41)
8+8 (37)
8+4 (37)
7+3 (37)
6+4 (34)
6+5 (32)
9+9 (29)
5+3 (26)
7+2 (24)
2+1 (21)
7+7 (20)
6+6 (20)
5+4 (20)
6+3 (20)
7+1 (20)
6+2 (19)
8+1 (19)
3+1 (19)
4+3 (18)
3+2 (17)
6+1 (17)
1+1 (17)
4+2 (16)
9+2 (15)
5+1 (15)
4+1 (15)
5+2 (13)
9+1 (13)
8+2 (13)
5+5 (9)
2+2 (9)
4+4 (8)
3+3 (8)

It is hard to believe that this table represents the real comparative difficulty of learning and retaining the addition combinations. Is it really more difficult to learn and remember $2 + 1 = 3$ than $5 + 4 = 9$; or $1 + 1 = 2$ than $9 + 2 = 11$, or even $5 + 2 = 7$? The writer has tried these combinations on many children below school age, as well as on many in the first and second grade; and barring cases of confusion which show nothing one way or the other, not once has this conclusion been verified. One little Miss of five puzzled long on $5 + 4$, and then: "Well, do you know how much $2 + 1$ is?" Promptly she said in a hurt tone, "Don't you think I know *anything?*" When children make mistakes on such combinations as $2 + 1$ or $1 + 1$, it is not because they do not *know.* There are many possible reasons: The combination may appear so easy that the attention is allowed to lapse for the moment, or the mistake may be one of those inevitable errors to which every human organism is liable. The most expert and conscientious computers check their most painstaking work; for they have learned by experience that, in spite of all, a cog is bound to slip now and then. In checking the summary of the table shown on page 200 the author found two errors in his own work and by mere accident he has detected more than one error in tabulations on this very subject published by very distinguished investigators whose names appear in this chapter.

Again there are certain errors which time is bound to cure so far as human error can be cured. Even if a six year old child should really have doubts about 2 + 1, there is no danger that such doubt will be shared by a normal boy or girl of sixteen. There will not even be the slightest hesitancy, though there may still be hesitancy about 9 + 7 or 8 + 5. Occasional errors there will always be; even the heart is said to miss a beat now and then.

C. L. Phelps studied the errors in 5950 papers from 238 eighth grade children. Following are the per cent errors (quoted from Starch):[1]

9+7 (3.32)
8+5 (3.10)
9+6 (2.60)
9+3 (2.55)
9+5 (2.50)
9+8 (2.44)
7+1 (2.42)
8+4 (2.38)
8+7 (2.30)
6+5 (2.27)
7+5 (2.25)
6+1 (2.22)
7+3 (2.02)
8+8 (1.98)
7+4 (1.95)
8+3 (1.94)
7+6 (1.56)
3+3 (1.46)
7+2 (1.32)
5+3 (1.31)
4+1 (1.30)
6+3 (1.19)
9+2 (1.16)
9+1 (1.09)
8+6 (1.04)
8+2 (.88)
5+2 (.86)
5+4 (.72)
5+1 (.72)
6+2 (.71)
6+4 (.69)
4+3 (.67)
2+1 (.64)
8+1 (.62)
0+8 (.62)
3+2 (.54)
0+5 (.52)
4+2 (.50)
3+1 (.45)
0+0 (.39)
2+2 (.37)
6+6 (.35)
9+9 (.30)
0+3 (.28)
1+1 (.27)
0+1 (.25)
0+4 (.24)
0+7 (.24)
0+9 (.15)
7+7 (.14)
0+2 (.14)
4+4 (.12)
5+5 (.07)
0+6 (.05)
9+4 (.)

It would be interesting to know, for instance, why there should be twelve times as many errors in 0 + 8 as 0 + 6, and why there should be five times as many in 6 + 1 as in 3 + 1. The facts would seem to indicate that factors quite distinct from knowledge or habits entered quite largely. It is not likely that these pupils were in the *habit* of adding 6 + 1 incorrectly. It would be interesting to know how many times each combination occurred and how the errors were distributed among the 238 children.

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[1] *Op. cit.* (p. 3), p. 414.

During the winter and summer of 1919 the following request was made of four different groups of teachers:

"Among the following 45 addition combinations, mark those 15 which in your opinion are the most difficult to learn and retain."

(Here appeared on the questionnaire 45 addition combinations.)

They were asked to put down their best personal judgment without consultation. There was no opportunity for consultation among the different groups. One group consisted of county superintendents in the State of Montana, one of teachers in Helena, Montana, one of teachers in Butte, Montana, and one of teachers attending the summer school in the State University of Montana.

The votes and ranks given by the various groups of voters and the total vote and rank given by all voters are shown by the following table:

COMBINATION	COUNTY SUPERINTENDENTS	HELENA TEACHERS	BUTTE TEACHERS	SUMMER SCHOOL TEACHERS	TOTAL
9+4	31 (1)	48 (2)	45 (2)	96 (1)	222 (1)
9+7	27 (7)	47 (4)	47 (1)	95 (3)	216 (2)
9+8	29 (4)	49 (1)	42 (7)	95 (4)	215 (3)
7+6	31 (2)	46 (5)	42 (8)	96 (2)	215 (4)
8+7	26 (8)	48 (3)	43 (4)	92 (5)	209 (5)
8+5	31 (3)	45 (7)	41 (9)	88 (6)	205 (6)
9+6	26 (9)	46 (6)	45 (3)	87 (7)	204 (7)
7+4	28 (6)	43 (9)	39 (10)	86 (8)	196 (8)
8+3	29 (5)	43 (10)	43 (5)	81 (10)	196 (9)
9+5	25 (10)	45 (8)	43 (6)	82 (9)	195 (10)
7+5	14 (15)	40 (11)	33 (12)	68 (11)	155 (11)
8+6	10 (18)	40 (12)	38 (11)	60 (13)	148 (12)
6+5	19 (11)	32 (14)	27 (14)	64 (12)	142 (13)
9+3	14 (14)	33 (13)	29 (13)	57 (13)	133 (14)
9+2	15 (13)	27 (15)	23 (15)	55 (14)	120 (15)
5+4	17 (12)	13 (17)	23 (16)	44 (15)	97 (16)

COMBINATION	COUNTY SUPERINTENDENTS	HELENA TEACHERS	BUTTE TEACHERS	SUMMER SCHOOL TEACHERS	TOTAL
8+4	6 (22)	25 (16)	18 (18)	36 (16)	85 (17)
7+2	11 (17)	8 (20)	22 (17)	33 (17)	74 (18)
4+3	12 (16)	11 (18)	18 (19)	28 (18)	69 (19)
6+3	6 (23)	11 (19)	12 (22)	26 (19)	55 (20)
7+3	8 (20)	6 (22)	13 (21)	23 (20)	50 (21)
5+3	9 (19)	7 (21)	17 (20)	17 (22)	50 (22)
5+2	8 (21)	4 (25)	12 (23)	17 (23)	41 (23)
9+9	4 (24)	6 (23)	9 (32)	21 (21)	31 (24)
3+2	3 (25)	3 (27)	9 (24)	8 (25)	23 (25)
8+8	1 (28)	4 (26)	5 (25)	10 (24)	20 (26)
6+4	0 (31)	6 (24)	4 (26)	7 (26)	17 (27)
7+7	0 (32)	1 (30)	2 (28)	6 (27)	9 (28)
8+2	2 (27)	2 (28)	2 (24)	2 (32)	8 (29)
7+1	1 (29)	0 (33)	3 (27)	1 (36)	5 (30)
6+6	0 (33)	2 (29)	0 (38)	3 (29)	5 (31)
6+1	3 (26)	0 (34)	1 (31)	1 (37)	5 (32)
6+2	0 (34)	1 (30)	1 (32)	2 (33)	4 (33)
4+2	0 (35)	1 (32)	0 (39)	3 (30)	4 (34)
8+1	0 (36)	0 (35)	0 (40)	3 (31)	3 (35)
9+1	1 (30)	0 (36)	1 (33)	1 (38)	3 (36)
5+5	0 (37)	0 (37)	2 (30)	1 (39)	3 (37)
3+3	0 (38)	0 (38)	0 (41)	2 (34)	2 (38)
4+1	0 (39)	0 (39)	1 (34)	1 (40)	2 (39)
2+1	0 (40)	0 (40)	0 (42)	2 (35)	2 (40)
3+1	0 (41)	0 (41)	1 (35)	0 (41)	1 (41)
5+1	0 (42)	0 (42)	1 (36)	0 (42)	1 (42)
1+1	1 (43)	0 (43)	0 (43)	0 (43)	1 (43)
4+4	0 (44)	0 (44)	0 (44)	0 (44)	0 (44)
2+2	0 (45)	0 (45)	0 (45)	0 (45)	0 (45)

This table shows a remarkable unanimity among these teachers. All four groups agree exactly as to which ten combinations are the most difficult, though they do not agree as to the order among these ten. Three of the four groups agree exactly as to the fifteen most difficult combinations and the fourth group places one of these fifteen as number eighteen, and places number sixteen in the total vote as number twelve. When one considers the many difficulties that beset the problem of testing this question by experimentation, it would seem that a combined judgment of practical teachers is as significant as the results obtained by counting errors in children's papers.

107. Comparative difficulty of multiplication combinations. — In the monograph referred to on page 197 Holloway obtained from the papers of 1215 third grade students the following order as to difficulty among the fundamental multiplication facts. Holloway used the table up to 12 × 12, but we have omitted from his list all combinations above 9 × 9. The numbers in parentheses indicate the number of errors.

9 × 7 (455)
8 × 7 (435)
9 × 8 (422)
9 × 6 (390)
8 × 8 (361)
8 × 6 (342)
9 × 4 (292)
7 × 6 (285)
7 × 7 (268)
9 × 9 (263)
8 × 4 (235)
7 × 4 (192)
7 × 5 (181)
9 × 3 (169)
9 × 5 (168)
8 × 3 (151)
6 × 5 (138)
8 × 5 (137)
6 × 4 (133)
6 × 6 (129)
6 × 3 (102)
4 × 4 (78)
4 × 3 (76)
7 × 3 (71)
8 × 2 (58)
5 × 4 (55)
6 × 2 (50)
5 × 3 (46)
1 × 1 (41)
9 × 2 (39)
7 × 2 (38)
5 × 5 (34)
4 × 2 (32)
4 × 1 (31)
3 × 1 (28)
5 × 2 (26)
2 × 3 (25)
9 × 1 (22)
3 × 2 (21)
7 × 1 (21)
6 × 1 (21)
5 × 1 (20)
2 × 1 (20)
2 × 2 (18)
8 × 1 (18)

The table given below was obtained in the same manner and from the same group as that of pages 200, 201.

COMBINATION	COUNTY SUPERINTENDENTS	HELENA TEACHERS	BUTTE TEACHERS	SUMMER SCHOOL TEACHERS	TOTAL
9 × 6	31 (1)	49 (1)	32 (1)	98 (1)	210 (1)
9 × 7	30 (4)	48 (2)	31 (2)	98 (2)	207 (2)
7 × 6	31 (2)	47 (3)	29 (5)	92 (5)	199 (3)
9 × 8	29 (6)	45 (6)	29 (6)	93 (3)	196 (4)
9 × 3	28 (7)	44 (7)	30 (3)	93 (4)	195 (5)
8 × 7	30 (5)	46 (4)	30 (4)	88 (8)	194 (6)

Continued:

COMBINATION	COUNTY SUPERINTENDENTS	HELENA TEACHERS	BUTTE TEACHERS	SUMMER SCHOOL TEACHERS	TOTAL
7 × 4	31 (3)	46 (5)	25 (8)	90 (7)	192 (7)
9 × 4	22 (9)	44 (8)	27 (7)	92 (6)	185 (8)
7 × 3	24 (8)	25 (13)	25 (9)	75 (9)	149 (9)
8 × 4	18 (13)	38 (10)	25 (10)	65 (11)	146 (10)
8 × 6	20 (10)	39 (9)	21 (11)	66 (10)	146 (11)
8 × 3	20 (11)	37 (11)	16 (14)	63 (12)	136 (12)
9 × 5	14 (16)	35 (12)	19 (13)	55 (14)	123 (13)
7 × 7	19 (12)	25 (14)	12 (18)	65 (13)	121 (14)
9 × 9	16 (15)	23 (15)	12 (19)	55 (15)	106 (15)
8 × 8	18 (14)	23 (16)	14 (16)	51 (16)	106 (16)
7 × 5	12 (17)	20 (17)	20 (12)	47 (17)	99 (17)
9 × 2	9 (18)	12 (21)	9 (21)	46 (18)	76 (18)
6 × 3	7 (20)	15 (19)	12 (20)	38 (19)	72 (19)
6 × 4	3 (24)	18 (18)	13 (17)	26 (21)	60 (20)
4 × 3	6 (21)	7 (22)	15 (15)	22 (22)	58 (21)
7 × 2	9 (19)	4 (23)	5 (22)	29 (20)	47 (22)
8 × 5	6 (22)	15 (20)	5 (23)	16 (23)	42 (23)
6 × 5	2 (25)	4 (24)	3 (25)	12 (24)	21 (24)
8 × 2	2 (26)	3 (26)	4 (24)	10 (25)	19 (25)
5 × 4	4 (23)	1 (30)	1 (35)	10 (26)	16 (26)
1 × 1	0 (33)	3 (27)	2 (26)	9 (27)	14 (27)
6 × 6	1 (29)	4 (25)	1 (36)	7 (28)	13 (28)
3 × 2	1 (30)	3 (28)	0 (38)	4 (29)	8 (29)
6 × 2	1 (30)	2 (29)	2 (27)	2 (33)	7 (30)
5 × 3	2 (28)	1 (31)	0 (39)	3 (31)	6 (31)
7 × 1	0 (34)	0 (34)	2 (28)	4 (30)	6 (32)
3 × 3	2 (26)	0 (43)	0 (43)	3 (32)	5 (33)
5 × 2	1 (32)	1 (32)	0 (40)	2 (34)	4 (34)
5 × 1	0 (35)	1 (33)	2 (29)	0 (38)	3 (35)
3 × 1	0 (36)	0 (35)	2 (30)	1 (36)	3 (36)
4 × 4	0 (37)	0 (36)	0 (40)	2 (35)	2 (37)
9 × 1	0 (38)	0 (37)	2 (31)	0 (39)	2 (38)
6 × 1	0 (39)	0 (38)	2 (32)	0 (40)	2 (39)
4 × 1	0 (40)	0 (39)	2 (33)	0 (40)	2 (40)
2 × 1	0 (41)	0 (40)	2 (34)	0 (42)	2 (41)
5 × 5	0 (42)	0 (41)	0 (42)	0 (37)	1 (42)
4 × 2	0 (42)	0 (42)	1 (37)	0 (43)	1 (43)
2 × 2	0 (44)	0 (44)	0 (44)	0 (44)	0 (44)
8 × 1	0 (45)	0 (45)	0 (45)	0 (45)	0 (45)

The table (pages 202, 203) shows remarkable agreement among these groups of teachers as to which combinations in the multiplication table are the most difficult. Among the ten most difficult, according to the combined vote, eight are included by all groups and no group places one of these lower than thirteenth. Among the first fifteen, according to the combined vote, twelve are included by all groups and no one is placed lower than nineteenth. Among the first twenty, seventeen are included by all groups and no one is placed lower than twenty-fourth. If these votes show nothing else, they certainly do show that the judgments of these groups of teachers are in surprising accord on this matter.

108. The child's individual list of difficult combinations. — For practical purposes, the most effective method of dealing with particularly difficult combinations is to obtain a list for each child of the combinations which are most difficult for him. In the Teachers' Edition of *Applied Arithmetics*, p. 317, Book I, and p. 301, Book II,[1] detailed suggestions are given for obtaining such lists with little trouble on the part of the teacher. Each pupil can practically make out his own list from drills given in the class. Children like to make such lists and take real delight in conquering the troublesome combinations which are thus revealed. It is found, usually to the child's surprise, that there are only a few that cause trouble and that he can end it entirely by concentrating on them. Before making the list, he has a general feeling that he "does not know his tables"; the trouble seems to be common to all of them, and little progress is made.

Recently the writer had occasion to help a boy who was in this predicament. The boy could never get a perfect record in his number tests; he always made a few mistakes. He estimated that by hard work he might be able to learn all the addition combinations perfectly in two weeks. By consulting several of his tests a list was made of the troublesome combinations. There were seven of them. On seeing this list the boy's feeling of discouragement was removed, and he concluded that he could master the tables in one day, which he did. Within twenty-four hours, he knew all the addition combinations perfectly; and since then, his test papers have usually been perfect with a stray blunder creeping in now and then. The good effects of such devices are seldom overestimated.

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[1] Lennes and Jenkins, *op. cit.* (p. 150).

109. Interchange of addends. — There has been much discussion as to whether a number combination such as $5 + 7$ should be habituated as one form or as the two separate forms $5 + 7$ and $7 + 5$. By actual tests it has been found that children who are drilled on one set of forms exclusively make very many more mistakes in the same combinations when the order is inverted than they do when the combinations are presented in the accustomed order. So it is argued that $5 + 7 = 12$ does not carry over to $7 + 5 = 12$.[1] It is no doubt true that it is possible to drill on $3 + 4$, $5 + 6$, $4 + 7$ in such a manner that $4 + 3$, $6 + 5$, $7 + 4$ will seem to the child new and strange combinations; there is almost no limit to the stupidity with which teaching can be done. On the other hand, it is perfectly possible so to develop these combinations and to drill on them in such a manner that perfect knowledge of both $5 + 7$ and $7 + 5$ may be acquired with very little more time and energy than of either of these alone.

The child learns by actual counting that $2 + 3 = 3 + 2$; $3 + 4 = 4 + 3$, and so on. After a few such pairs of facts are learned, the child is led to infer others directly, *provided* his attention has been called to this fact in such manner as to lead him to develop the *idea* that "the sum is the same no matter what the order is." The teacher should be aware that she is developing one of the fundamental principles of mathematics — the "commutative law of addends." In the early part of number work such pairs of number facts should be developed simultaneously. Thus a line of drills in addition may look like this:

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[1] "Drill was given on the combinations 6 plus 8, 7 plus 6, 9 plus 8, 6 plus 7, 9 plus 6, in these orders. Tests were then given on these combinations in these orders and also in reverse orders. The series given for the practised orders were 97.21 per cent correct while in the reverse orders they were 72.31 per cent correct." Holloway, *op. cit.* (p. 197), p. 21.

3	4	5	7	6	8	7	8	4	7
4	3	7	5	8	6	8	7	7	4
—	—	—	—	—	—	—	—	—	—

Ultimately the child will react, not to any particular form of 3 + 4, but to "3 and 4" to be taken together in addition and the *order* in which they are said or written will scarcely be attended to. A little girl sees two children coming to play and pays no attention to their relative position at the time she sees them. She runs to her brother and says, "John and Mary are coming to play." The order of the names has no significance for her; she tells her mother that "Mary and John" are coming. There is no reason to believe that the situation with the order of addends is inherently different; but there is of course no way of predicting what abnormalities may be developed artificially. Persistent drill in saying "John and Mary" in precisely this order would no doubt make "Mary and John" sound strange and for the moment absorb the attention sufficiently to impede other mental activities (for example, thinking of the sum of 5 and 7 with the same rapidity and accuracy that this could otherwise be done).

These remarks apply, of course, to the order of factors in multiplication. Experience with learning children makes it absolutely certain that, by pointing out such relations as $2 \times 3 = 3 \times 2$ from the beginning and associating such pairs of products persistently, the idea develops that the order makes no difference and gradually the two forms 2×3 and 3×2 melt into one, and either calls up the response 6 with equal rapidity and certainty.

110. The subtraction combinations. — The method of studying the subtraction combinations depends upon the general point of view adopted in regard to subtraction. If subtraction is regarded primarily as "taking away," then all the subtraction combinations must be learned separately. At least that is the way it works itself out with the child. If the child is trying to find $7 - 3 = 4$ by using objects, he starts with 7 objects, then removes 3 of them, and finds there are 4 left. In the actual process of learning, this does not connect itself up naturally, if at all, with the addition combination $3 + 4 = 7$. In current teaching, the two arise so separately that testing subtraction by adding the remainder to the subtrahend is entirely strange to the child. If subtraction is regarded as a taking-away process, there is nothing to do, therefore, but to learn each subtraction combination as the addition combinations were learned.[1]

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[1] "Ballard gave tests in fundamentals to 71 English schools of which 23 had been taught subtraction by the method of equal addition while the rest had been taught it by the method of decomposition. While there was little difference in the average ability of the two groups in the other three fundamental operations, there was a very striking superiority in the score for subtraction of the equal addition group. At 13 years of age it amounted to over 10 per cent and at earlier ages it amounted to over 40 per cent. Inasmuch as the decomposition, or less efficient, method is the one in general use in this country, it is evident that this matter deserves careful attention." Starch, *op. cit.* (p. 3), p. 407.

"Mead and Sears performed two experiments in comparative economy of methods in arithmetic. The first was to compare the efficiency of the ordinary 'take away' method of subtraction which involves the learning of an entire subtraction table, as compared with 'addition' subtraction which permits the use of the addition table, thus saving the learning of an entire table. In the first the formula is '8 minus 2 equals what?'; in the second, '2 plus what equals 8?'. Two second grade classes of approximately equal median ability as indicated by the Courtis tests were taught subtraction by the respective methods thirty minutes per day for four months, all other factors being equalized as fully as possible. Tests given periodically throughout the experiment showed that at the outset the addition method was superior but the 'take away' method gradually overtook it until at the end of the four months the 'take away' method was superior by 4.5 points which was nearly one third of the final median score made by the addition group. This difference disappeared, however, when both groups were tested on longer examples." Starch, *ibid.*, p. 408.

If, on the other hand, subtraction is regarded as finding how much must be added to one number to make the sum equal to another number, then the subtraction combinations become simply a repetition of the addition combination. Thus when subtracting 3 from 7 is regarded as finding the missing number in $7 = 3 + ?$, the required number 4 arises at once from the known combination $7 = 3 + 4$. This second point of view leads directly to the so-called Austrian method of subtraction which is considered on page 231. If it is decided to adopt this method, the elementary subtraction problem should be read in the form of $7 = 3 + ?$ and not in the form $7 - 3 = ?$. By using this form consistently, nearly all the time used in memorizing the subtraction combinations may be saved, while at the same time the grip on the addition combinations is strengthened. Even when the examples in subtraction are written in the form:

$$\begin{array}{r r} & 7 \\ \text{subtract} & \underline{3} \end{array}$$

they should be read $7 = 3 + 4$ and the 4 written below the line.

It should, of course, be made clear to the child that subtracting 3 from 7 may be regarded as doing any one of the following four things:

(1) finding how much must be added to 3 to make 7
(2) finding how much greater 7 is than 3
(3) finding how much smaller 3 is than 7
(4) finding how much is left when 3 is taken away from 7.

The teacher should notice that the Austrian method connects more directly and naturally with the first three of these and that indeed the fourth may be so connected very readily and without elaborate explanation. It appears, therefore, that the Austrian method of subtraction is more directly and naturally connected with the demands of practical life than is the other. Of course, the usual method of making change, which comprises more than half of all subtracting actually done in practice, leads directly to the Austrian method.

111. Objections to the Austrian method — inhospitality to what is new. — This topic may merit a separate paragraph. The inertia which new things have to overcome is of greater consequence than is commonly supposed. A story will serve to illustrate: Some twenty years ago and more, I rode much on a bicycle, even riding several "centuries" each season. Then the "wheel" went out of fashion, and I rode no more. However, while in Europe a dozen years later, I found myself obliged to ride again. The only style of wheel available had a coaster brake, to which I was not accustomed. How I wished for the good old-fashioned bicycle! To get used to the new wheel cost me a week of real work — of perspiration and pain and fatigue. I held a purely intellectual belief that the new style of wheel was superior and that all that was necessary was to get used to it. In actual practice, however, I had to struggle constantly with the fact that I could ride the old wheel faster, easier, and in every way more comfortably than the improved new style. The fact that I was obliged to ride and had no other wheel to ride saved the day. In time I got so I could ride the new wheel as easily as the old and, finally, so I could ride it more easily. The old habit-paths in my nervous system which had been built earlier in life had now been broken down, so far as was necessary, and new ones fitted to the new-fangled wheel.

How many excellent devices are left unused because of this dislike to learn new tricks! One of the blessings of a competitive system is that those who compete are driven to adopt new ideas, new methods, and new machinery on pain of industrial extermination. No less drastic alternatives would seem sufficient to cause those well on in life to break old habits and build new ones. In estimating the comparative merits of the two methods of subtraction, discussed above, this universal fact is sure to make itself felt. The writer learned to subtract by "taking away," "borrowing," etc. It cost him no little effort to learn the other and to habituate himself to it. It is not urged that children who have already learned one method of subtracting should abandon it and learn another. The teacher, however, occupies a peculiar position. It is the absolute right of the child to be taught the best methods that are known, even though it does cause the teacher some temporary trouble and annoyance.

112. Further objections to the Austrian method — group inertia. — Perhaps the most important objection raised against the Austrian method of subtraction is that children ask their parents to help them, that the parents in most cases are using the older method, and, hence, not only does confusion arise but in addition the parents acquire a new objection to the school, which too often is regarded as a breeding place of silly and newfangled notions, sadly lacking in common sense. In some cases, this argument may be of sufficient might to cause the wise abandonment of the Austrian method. In some localities the schools are strongly intrenched and people have learned to have faith in the wisdom of those who have them in charge. In other localities the word "schoolma'am" (applied to male and female alike) is spoken with a mixture of disdain and toleration. The first assumption is that real sense and wisdom have been reserved for those who are in "practical business." In such localities it may be well enough to defer to local prejudice and use the older method. This inertia of the group caused by imposing upon each new generation the customs and habits of the one that precedes is, of course, of great value as a conservator of what has been found safest and best. Thus wisdom accumulates. But it is also a great and powerful obstacle to the general adoption of new and sometimes highly important improvements.

113. The multiplication table and related number facts. — The actual learning of the multiplication table involves pretty much the same principles and devices as the learning of the addition table. There are, however, certain number facts, specially related to each fact of the multiplication table, which require study at this point. At the outset $3 \times 4 = 12$ and $4 \times 3 = 12$ are very different facts to the child. Knowing that three 4's make 12 does not at all imply to the child that four 3's make 12. This should be made clear not by the teacher's pronunciamento but by experience with "things." However, after the child has established a few pairs of facts, such as $3 \times 2 = 6$, $2 \times 3 = 6$; $2 \times 4 = 8$, $4 \times 2 = 8$; etc., he finds a basis for the belief that the numbers whose product is found may always be interchanged in order. The wise teacher does not make the concrete experimental basis too extended. The child is willing enough to generalize on a slender basis and there is no reason for making it unduly broad at this point.

The child should not go far in learning the multiplication table before a fact like $6 \times 7 = 42$ carries with it automatically the related fact $7 \times 6 = 42$. Nor is this all: As the child proceeds in learning the multiplication table, he should be taught the meaning of such statements as $12 \div 3 = 4$, $12 \div 4 = 3$, $1/3$ of $12 = 4$, $1/4$ of $12 = 3$, and he should be taught (not simply told) that all these facts *follow* from the fundamental fact $3 \times 4 = 12$. If this work is done properly, each new fact learned in the multiplication table will bring with it, without further effort or study, usually five related number facts. Thus, $6 \times 7 = 42$ will bring with it without further effort the facts, $7 \times 6 = 42$, $1/6$ of $42 = 7$, $1/7$ of $42 = 6$, $42 \div 6 = 7$, $42 \div 7 = 6$. Indeed, these statements will soon be seen to repeat the same fundamental fact. The points of view which give rise to the various statements are different but the object observed is the same. It does not take long to bring home to the child that every combination in the multiplication table may be looked at from these various points of view.

In this manner division and fractions within the tables are learned along with the multiplication table and with very little greater total effort than would be required to learn the table alone. Indeed the total effort may even be less, for these forms afford new ways of stating the multiplication facts, and the comparatively interesting form of the repetition may cause the child to learn the table itself more readily than he would if products only were used.

Through a park there is a main path and many smaller ones leading in all directions. One child is guided through this park by a systematic nurse, who daily follows the main path in the same direction. From day to day the paths, trees, ponds, and surrounding neighborhood present themselves from exactly the same angles. Another child is allowed to enter the park at various points, to travel the main path in either direction, and to explore inviting bypaths. This second child not only has a richer and more satisfying experience — not only does he learn more about the park in general, but he acquires a much firmer grasp of the "geography" of the main path. The totality of his experience melts into a complete picture in which the main path stands out clearly in all its relations to the rest. The child who follows the path daily in the same direction and makes no side excursions has a scant and uninteresting setting for the picture of his path, a picture which will soon fade from memory. The application to the matter in hand is obvious.

114. Making lists of combinations already learned. — In learning the addition and multiplication tables, a list should be made, possibly on the blackboard, of all the new facts learned. Each new fact should be especially pointed out as *new*. This will serve to direct particular attention to it as it is introduced and to make easier its retention in the memory. If a list is kept on the board of the combinations already learned (but not of the results), it may be made the subject of occasional drills, and the child will acquire a feeling of responsibility for facts contained in it. It becomes a disgrace to make a mistake in these combinations. These the child feels he *must* know. Further, such a list shows how few combinations there are in all and serves to create a feeling that, instead of an endless chaos which the child is expected to put in order, there are comparatively few facts to be memorized. These few he will resolve to master. This is particularly true when a very small number of new combinations are introduced at each step in advance.

115. Mastering combinations separately. — Experimental evidence and common sense both lead to the conclusion that the fundamental addition and multiplication combinations should be mastered separately and not in serial tables.[1] In practice each combination arises as a part of a more complicated process and entirely dissociated from a table. The aim is to learn the combinations in such manner that each one may be recalled automatically while the mind is absorbed with a larger process. The ability of recall must be built into the hierarchy of abilities necessary for the performance of the fundamental operations. The proper procedure would seem to be: (1) master the combinations, say $4 + 7$ or 4×7, separately without any reference to a table of 4's or 7's; (2) practice using the combination as a constituent part of the larger operation of practical addition or multiplication. There is some value in looking over a table; it helps to *place* each combination in a setting which throws light on its essential nature. The equations, $5 \times 7 = 35$, $6 \times 7 = 42$, etc., tell something about the nature of products and may even afford a clue that will help in immediate and direct recall. The thing to guard against is learning the tables by *rote* and a silent repetition of a table each time a combination is needed. Drill under pressure will serve admirably to break up this habit even when once formed.

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[1] "Kirkpatrick (quoted by Starch) tested the relative efficiency of learning tables by rote and of learning them by writing down products in miscellaneous order, and found an advantage for the latter method." Starch, *op. cit.* (p. 3), p. 406.

116. Order of introducing number combinations. — It seems to be agreed fairly generally that subtraction combinations should be introduced along with addition, and division combinations along with multiplication. Even this statement requires qualification. What really takes place is about as follows: A few of the simplest addition combinations are first introduced without any reference to subtraction — enough to develop the idea of sum and to obtain acquaintance with the symbols + and =. Supplying missing numbers in such expressions as $6 = 4 + ?$ and questions, such as "Mary had four apples and gave Jane two; how many had she left?", precede the use of the word "subtraction." The idea of subtraction is then developed and also the relation between addition and subtraction. From this point on, addition and subtraction combinations are developed simultaneously, the latter being regarded simply as a slightly different phase of the former. Similar remarks apply to the introduction of multiplication and division combinations, the latter also being expressed by means of fractions. Thus, $42 \div 6$ and $1/6$ of 42 are regarded as the same problem.

There is less unanimity of opinion as to the point where products should be introduced. Should all the addition facts be learned before any products are taken up? If not, at what point in the development of addition should products first be considered, and how rapidly should they be introduced as the learning of addition proceeds? As indicated by published curricula and school texts, the most generally accepted practice seems to be: After the ideas of adding and subtracting have been developed, counting by two's and three's is introduced, and along with it, "two times two," "two times three," etc. By the time all the fundamental addition facts have been learned, multiplication combinations with products as great as 20 or possibly 25 have usually been introduced. There is often an exception in the table of 5's which may be learned up to 9 × 5 or 10 × 5. The rest of the multiplication table is usually not taken up until the processes of addition and subtraction have been completed.

There is no doubt good reason for this practice. The learning of simple combinations, such as 3 × 2 or 4 × 3, is intimately associated with addition. The child passes from $2 + 2 + 2$ directly to 3 × 2. In the case of the larger products, this connection is not so intimate, and multiplication becomes a more independent process. The full development of the forms associated with each multiplication fact (see section 113) may well be delayed until these are studied more systematically after the addition and subtraction processes have been mastered. (Note the distinction between the fundamental combinations and the *processes* of arithmetic.)

117. The necessity and character of drill on fundamental combinations. — There has been great divergence of opinion as to the necessity of drill on the fundamental operations. At the one extreme in the range of opinion are those who believe in little or no drill either on the combinations or the fundamental operations. With them all practice would come in connection with the solving of concrete problems. Another type of opinion is represented by those who believe that all drill should be on the fundamental operations and none on the combinations themselves. Thus, no drill would be given on the addition combinations as such, but much or little drill might be given in abstract column addition. Still others would have drill on the separate combinations as well as on the fundamental operations. All possible shades of opinion are found ranging from one extreme to the other.

At the present time, there seems to be a growing consensus of belief that considerable drill in the combinations is necessary. It is argued, the author believes rightly, that the ability to solve a concrete problem is a highly composite ability (see section 10) and that each component ability should at times be sequestered for separate drill. The efficiency experts in industry are attaining their ends by analyzing composite operations (laying bricks, for instance) into many constituent simple operations and drilling on each of these and then again practicing on the complete operation. In this manner striking results have been achieved in many cases. The analogy with the question at issue here would seem to be perfect. On the one hand, it is not sufficient to develop the number facts in connection with concrete material, though this is necessary. Ultimately, the numbers and their combinations must attain existence in the mind of the child as abstract entities; only in this form will they become instruments for ready use in the manifold applications where they are needed. It is not "4 boys and 5 boys are 9 boys" but "4 + 5 = 9" that is needed. On the other hand, learning the combinations by using them as parts of more complicated processes fails to single out the troublesome ones for separate attention (see section 108). In the study of bricklaying, it has been found that one person has trouble with one element of the operation and others with other elements. No doubt there are other factors entering as well, which make this course inadvisable. It is concluded, therefore, that separate drilling on the fundamental combinations is necessary — a good deal of it at the time they are learned and later occasional drills from time to time over a considerable period.

In character, the drills should be snappy and should be so motivated that the child will have a keen desire to improve both in accuracy and in speed. The importance of a strong effort to learn on the part of the child has become increasingly evident from the work of the experimental psychologist and is one of the really substantial results thus far obtained in the laboratory. It is for this reason that proper motivation of drill is so important.

118. Special devices. — Many special devices have been noted in this chapter and in Chapter III. Some others will be noted here. Every good textbook contains a great many. All teachers are agreed on the importance of play, but perhaps not all are agreed as to *how* important it is. Qualitatively, we are all agreed but, perhaps, not quantitatively. A girl climbing a number ladder dreamed along in a leisurely fashion, making mistakes not infrequently, until her older brother said, "If you fall off that ladder, you will bust your head." There was no more carelessness. The child did not want to "bust her head." No amount of persuading or scolding would have worked the change. Shortly afterwards the boy was trying his hand at a series of exercises printed in ordinary form. Presently he conceived of each combination as a brick placed in a puddle, and started to run, stepping from brick to brick without getting wet. These "play situations" were of great moment to the little folks. They made them work as few other motives or devices would. The alert teacher who enters into the spirit of child life will discover numberless situations of this kind as the work progresses.

One little boy may need mothering when he comes with his troubles, but to another it is not only safe but important to say: "You can't do that! Of course you can. Now you just go right ahead and do it. When you get it done, let me see it." The right kind of a child goes back to his seat muttering to himself, "I will show her all right." What hope for future years is there in such mutterings! One youngster may be greatly helped by being told to help someone else, though he himself scarcely knows that in which he is to give help. By trying to help another perhaps a little more backward than himself, his attention is made to dwell more persistently on the subject than it otherwise would, and this may be the very means of clearing it up for both the helper and the helped.

119. Applications. — The number system and its combinations should be built into the child's life as he learns them. This is not so much for the purpose of motivating the work as for the purpose of making the number facts organic parts of the child's mentality. For one thing, this makes forgetting less likely. Facts on which we have fastened many and diverse strings of associations are thereby insured against loss. For another thing, it will form a small part of an element in education which should be extended as far as possible over its whole extent: namely, the seeing of the human significance of what is done.

There is a tremendous difference between the "educated fool" and the sane, level-headed man or woman with a first-rate education. If there is one thing which educators should keep in mind, it is the danger of turning perhaps our most promising material into educated fools, and the one great preventative is to look for as broad human significance as is practicable in everything that is studied. For the child learning the addition and multiplication tables, this consists in using the number combinations in such situations of life as naturally come within his knowledge and interests. Texts in elementary arithmetic furnish such applications in abundance. The purpose here is simply to point out the deepest reason for introducing them. Unless the teacher comprehends this reason in its full significance, she will fail partly, no matter how excellent the textbook and the course of study may be. Another type of applications, namely, the use of the combinations in the more complex process of performing the fundamental operations, will be considered in the next chapter.

REVIEW QUESTIONS

1. Discuss the method by which the child obtains his first knowledge of numbers. What is the distinction between counting and measuring? Discuss McLellan and Dewey's use of the word "measuring." (§ 102.)

2. Contrast the Grube method of teaching numbers with the method of developing numbers more rapidly as noted in section 103.

3. Discuss the question as to how far the number combinations should be derived from a concrete basis. (§ 104.)

4. What is meant by "making number combinations automatic"? Discuss the necessity of reviews and their varied applications. (§ 105.)

5. Discuss the comparative difficulty among the addition combinations. Is there any reason why 7+1 should be more difficult than 8+7 and why 9+4 should be the least difficult of all the combinations? (See list from Phelps on page 199.) What reasons would you assign for some of the peculiarities in Phelps's list? (§ 106.)

6. Among the fifteen most difficult multiplication combinations as given on page 200, how many are included among the fifteen most difficult according to Holloway (see page 198)?

7. Discuss the importance of obtaining for each child a list of those combinations which are most difficult for him. Describe a convenient method for obtaining such lists. (§ 108.)

8. Discuss the importance of presenting the members in such combinations as 5+7 in varying order. Also discuss this for combinations such as 5×7. (§ 109.)

9. Describe the additive method of subtraction. Show why, in using this method, we should say, for 12−7=5, "twelve equals seven plus five" and not "seven plus five equals twelve." Give examples in subtraction in which the taking-away method is more natural; also give examples in which the additive method is more natural. (§ 110.)

10. Discuss various objections to the additive, or Austrian, method of subtraction. (§§ 111, 112.)

11. What number facts are naturally related to a combination, such as 7×8=56? Discuss the economy of teaching all these in connection with this combination. (§ 113.)

12. Discuss the value of keeping a list of the number combinations already learned and adding to it each new combination as it is mastered. What is meant by "mastering each combination separately"? (§§ 114, 115.)

13. Some say that little or no drill is needed on the fundamental combinations while others would continue occasional drills on them for some time. Discuss these two views. (§ 117.)

14. Discuss applications of the number system and its combinations, (a) as a motivating force; (b) as insuring against forgetting; (c) as a fundamental aim in education. (§§ 84, 100, 119.)

CHAPTER VII

Fundamental Operations on Whole Numbers

120. Addition by endings. — The whole process of column addition may be based on the 45 fundamental combinations and the so-called "addition by endings." From the relation, 3 + 4 = 7, the child may be led to see that 13 + 4 = 17, 23 + 4 = 27, 33 + 4 = 37, etc. These combinations should be written in columns on the board side by side, as shown below; and initially they should be illustrated by the use of marks, sticks, or any other objects that may be convenient for counting.

$$\begin{array}{cccccc} 3 & 13 & 23 & 33 & 43 & 53 \\ \underline{4} & \underline{4} & \underline{4} & \underline{4} & \underline{4} & \underline{4} \\ 7 & & & & & \end{array}$$

The first combination, 3 + 4, thus becomes a "key" to the others. This "key" combination will carry over to the higher decades, *provided* the idea is developed, otherwise not.[1] That is, the child may know perfectly well that 3 + 4 = 7 and at the same time fail to respond to "83 + 4 = ?" if the relation between the two has not been pointed out to him. On the other hand there is not the slightest doubt that 3 + 4 = 7 will be a real help in 83 + 4 if the connection is developed.

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[1] The truth of this statement is not infrequently denied. Thus: "School practice neglects them [the laws of connection forming an association or habit] when it fancies that knowledge of the addition combinations in the higher decades (that is, 17 + 9, 23 + 5, 38 + 4, etc.) will come by magic after 7 + 9, 3 + 5, 8 + 4, etc. are once known." Thorndike, *op. cit.* (p. 21), Vol. II, p. 21.

They do not come by magic, unless the workings of the mind are magic, but they can easily be caused to come, for "I have seen them come." It is true that they do not necessarily come unless the idea is *developed*, and this simply shows that at this early age the child is not always able to develop this idea unaided (though some can, for again I have seen them do this too). The fact that in some cases this connection is never made simply shows how badly teaching may be done.

But perhaps the following excerpt represents Thorndike's present view more accurately: "And even for the most gifted child the formation of the connection '8 and 7 = 15' probably never quite insures the presence of the connections '38 and 7 = 45' and '18 and 7 = 25.' " *Op. cit.* (p. 42), p. 52.

When one series of such combinations has been studied the "key" combination in another series will carry over more readily, as for example in:

$$\begin{array}{ccccccc} 4 & 24 & 34 & 64 & 74 & 84 & 94 \\ \underline{5} & \underline{5} & \underline{5} & \underline{5} & \underline{5} & \underline{5} & \underline{5} \\ 9 & & & & & & \end{array}$$

In these two sets the sums in the "key" combinations are less than 10. All such combinations should be studied and ability to respond readily achieved before the others are taken up. The work should be varied by omitting the key combinations and the orders changed as in:

$$\begin{array}{ccccc} 22 & 47 & 56 & 62 & 74 \\ \underline{3} & \underline{2} & \underline{3} & \underline{5} & \underline{5} \end{array}$$

The next step is to develop such series as 7 + 6 = 13, 17 + 6 = 23, . . . Thus:

$$\begin{array}{ccccccccc} 7 & 17 & 27 & 37 & 47 & 57 & 67 & 77 & 87 \\ \underline{6} & \underline{6} & \underline{6} & \underline{6} & \underline{6} & \underline{6} & \underline{6} & \underline{6} & \underline{6} \\ 13 & & & & & & & & \end{array}$$

Initially, these should also be illustrated with objects and the idea made clear that the sum lies in the next decade. It should be pointed out that this is also the case in 7 + 6.

For the purpose of drill on adding by endings, the child may write out and paste on cardboard a square block of numbers containing all integers from 0 to 100, thus:

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
. .	. .	. .	. .	. .	. .	. .	. .	. .	. .

Drill in adding by endings may then be had by adding, say 6 to the first column, 3 to the second, 5 to the third, etc. All these involve additions of the type 3 + 4, 13 + 4, etc. When these are well in hand, drills like adding 8 to the fourth column, 7 to the sixth column, etc., should be repeated from time to time until satisfactory speed and accuracy are attained.

Later the work may be varied by having children give sums as the teacher names numbers as follows: 6 and 7, 36 and 7, 86 and 7, 26 and 7, 66 and 7, 46 and 7, 76 and 7, 56 and 7. Finally when the children can give such sums without hesitation, the key combinations, as 6 + 7 in this case, are omitted and mixed drills like the following are given: 43 and 8, 67 and 4, 43 and 5, etc. Much of this work may be mixed with single column addition, but it should all be done before "carrying" is taught, for otherwise the child will add 57 and 8 by adding 7 + 8 and carrying. In column addition carrying cannot be used. Adding by endings is useful solely as a preparation for column addition.[1]

This carrying over of the fundamental combinations to the higher decades by means of the *idea* of the process is an example of what takes place over and over again in all good teaching and especially in arithmetic. There is no surer indication of bad teaching and of bad learning than the development of a set of unrelated responses when it is perfectly possible so to relate them that they will be controlled by a more general idea or principle. General principles are the most economical elements in mental activity, and there is always waste wherever an opportunity to develop a principle is lost. The only limit that should be set to this work is the ability of the learner to master the principle. It is in such instances as the one given here that we may start to develop the habit of raising the work of the child to the conceptual level. It is all nonsense to say that "this would be implying the possibility of developing generalized habits and there are no generalized habits." We all know that some people's information is largely in the form of isolated facts while other people's is largely in the form of related facts. That is, some people are in the "habit" of grasping facts in relation while others are not. No doubt this habit can be developed if education is properly organized.

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[1] In some texts addition by endings is given after column addition and carrying are taught. This shows, as nothing else could, an absolute misunderstanding of the purpose of adding by endings.

Not only are facts learned more easily when seen in relation, but they are more surely remembered. Moreover, the general principle by means of which the facts are related is often more valuable than are all the unrelated facts even if retained in memory. Recently one of our great industrial organizations stated in one of its advertisements: "A good theory is the most practical thing in the world." In this case the advertiser stated a profound truth.

121. Column additions. — Short columns are, of course, added before the children have become perfect in adding by endings. It should be borne in mind, however, that adding by endings is the foundation of all column addition; and if the latter does not go well, adding by endings should be brushed up. There is indeed a certain chasm between adding by endings and column addition — there is always a chasm when a new whole is made out of parts which may be known ever so perfectly. For this reason patience must be exercised. Nevertheless, the connection should be insisted upon, and it will soon be found that the chasm has been bridged, the adding by endings has been adjusted to its new environment, and all will go well.

122. Carrying in addition. — In introducing carrying in addition, as in introducing other new elements in arithmetic, an effort should be made at the outset to rationalize it. The effort may not be entirely successful; probably it will not be. It will serve at least to give the child the feeling that there *is* an explanation, that the new element does not drop mysteriously from the clouds. No insistent attempt should be made to make the child understand the explanation fully. At any event, it will appear again in reviews in later years. Perhaps no fault is more common, especially with young, bright, enthusiastic teachers, than explaining and reasoning too much. The child is not capable of a sustained course of reasoning as understood by adults. Children learn in any way they can — mostly by imitation. Testing the correctness of the result is entirely satisfactory to them and gives complete confidence in the method. The adult's argument which shows that such and such a method *must* give the correct result is for a later day.[1]

It is probably best to confine the attempted explanation at this point to the carrying of tens. The carrying of hundreds can be done by analogy and the explanation omitted. That is, carrying might be explained in the example in the margin, but not in the one on the next page.

$$\begin{array}{r} 48\\ 69\\ 38\\ \underline{42}\\ 197 \end{array}$$

Drill on the nature of the decimal number system comes in as an incident in the attempted explanation: How many tens and how many ones in 48? in 76? etc.

Probably a majority of teachers will agree with the following statements, by Suzzallo:

> "Again the rational needs of a student of mathematics may require an understanding of the reasons why we 'carry' in column addition, but the effective everyday use demands an accurate *habit* of 'carrying' rather than an accurate *explanation*."[2]

---

[1] "Boys are not philosophers and logicians. Boys are usually exceedingly stupid in anything requiring concentrated reasoning. It is not in the nature of their soft brains that they should take kindly to Euclid and other stuff of that logic-chopping kind. But they usually possess another sort of mental ability — namely, the ready acquisition of new facts and ideas — and that is what should be taken advantage of. They have also the power of learning to work processes, long before their brains have acquired the power of understanding." Oliver Heaviside, Discussion on the Teaching of Mathematics, *op. cit.* (p. 60), p. 67.

[2] *Op. cit.* (p. 165), p. 11.

> "(1) Any fact or process which always recurs in an identical manner and occurs with sufficient frequency to be remembered ought not to be 'rationalized' [the writer would say: 'except when first introduced'] for the pupil but 'habituated.' The correct placing of partial products in the multiplication of two numbers of two or more figures is a specific case. (2) If a process does recur in the same manner but is so little used in after life that any formal method of solution would be forgotten, then the teacher should 'rationalize' it. (3) If a process always does occur in the same manner but with the frequency of its recurrence in doubt, the teacher should both 'habituate' and 'rationalize.' The division of a fraction by a fraction is frequently taught both 'mechanically' and by 'thinking it out.' (4) When a process or relation is likely to be expressed in a variable form, then the child must be taught to think through the relations involved and should not be permitted to treat it mechanically through a mere act of habit or memory."[1]

For the purpose of checking results by adding in the opposite direction, it is well to establish the habit of adding each column *downward* and then checking by adding *upward*. The work of checking is facilitated by writing the number to be carried directly under the number written into the sum as shown in the example in the margin.

$$\begin{array}{r} 816 \\ 249 \\ 216 \\ 782 \\ \underline{491} \\ 2554 \\ 22 \end{array}$$

123. Early use of large numbers. — In many schools large numbers are avoided as long as possible. It is urged that children do not "comprehend" such numbers. It is, of course, obvious that the fundamental operations must first be studied in connection with small numbers which can easily be illustrated by means of objects. It should not be forgotten, however, that children use large numbers readily and that the fundamental operations are mainly formal. After a child has learned to carry in addition, he will add three or four column numbers as easily as two column numbers. Seldom does the size of a number bother a child. He uses telephone numbers, house numbers, and

[1] Suzzallo, *op. cit.* (p. 165), p. 65.

license numbers on automobiles without serious trouble. Again, perhaps without "comprehending" the meaning of 20,000, he knows that a town with 20,000 people is twice as large as one with 10,000, and four times as large as one with 5,000. In the last analysis that is very nearly what the adult means by "comprehending" a large number.

There would seem, therefore, to be no objection to the use of large numbers in drills on the fundamental operations as soon as these have been fairly learned. All efforts to "image" large numbers must of necessity end in failure and should not be attempted. The comparing of large numbers among themselves is important and should be practiced until the relations among hundreds or among thousands are as readily described as the relations among units.

124. Subtraction: first method. — At the beginning of the child's number work one of the several methods of subtraction should be selected, and this method should be adhered to throughout the course. Nothing insures confusion more certainly than the indiscriminate use of different methods in different grades. The method described in this section may be called the "taking-away-borrowing-method." It will be recognized as the method which most of us learned in our school days.

In teaching any subject it is important so to arrange the successive steps that each element of difficulty is encountered separately. The series of examples given below shows how this may be done for subtraction. At the outset the work may be accompanied by oral or written statements as indicated.

First set of examples: No borrowing.

896	6 less 5 is 1. Write 1.
541	9 less 5 is 4. Write 4.
355	8 less 3 is 5. Write 5.

A little later attention is called to the fact that the numbers represent ones, tens, and hundreds. Then the oral or written statement may be as follows:

896	Ones: 6 less 5 is 1. Write 1.
355	Tens: 9 less 5 is 4. Write 4.
541	Hundreds: 8 less 3 is 5. Write 5.

It is important that some short form of this sort be used. Repetition of the words helps to clarify the idea and lays the foundation for fuller understanding later on.

Second set of examples: Two place numbers with borrowing; no zero in minuend.

Let it be required to subtract 56 from 84.

We cannot take 6 from 4 and so we "borrow" ten from the 8 tens, which added to the 4 makes 14. 6 from 14 is 8. Since one ten was borrowed from the 8 tens, there are left only 7 tens. 5 tens from 7 tens is 2 tens.

The written form may be as follows:

84	Ones: 14 less 6 is 8 (borrow).
56	Tens: 7 less 5 is 2.
28	

Third set of examples: Numbers containing three or more figures; no zero in minuend; one borrowing.

Let it be required to subtract 465 from 738. The written form may be as follows:

738	Ones: 8 less 5 is 3.
465	Tens: 13 less 6 is 7 (borrow).
273	Hundreds: 6 less 4 is 2.

The form is precisely the same whether one, ten, or one hundred is borrowed. Attempt at fuller explanation may bring more confusion than light.

Fourth set of examples: Numbers containing three or more figures; two or more borrowings; no zeros in minuend.

The written form may be as follows:

4174	Ones: 4 less 2 is 2.
2892	Tens: 17 less 9 is 8 (borrow).
1282	Hundreds: 10 less 8 is 2 (borrow).
	Thousands: 3 less 2 is 1.

Fifth set of examples: Last figure in minuend zero.

290	Ones: 10 less 8 is 2 (borrow).
158	Tens: 8 less 5 is 3.
132	Hundreds: 2 less 1 is 1.

Sixth set of examples: Borrowing across a zero.

9	
804	Ones: 14 less 7 is 7 (borrow).
267	Tens: 9 less 6 is 3.
537	Hundreds: 7 less 2 is 5.

Here we explain that we cannot borrow from zero as that has nothing to lend. Hence we borrow from the 8. When we borrow across a zero we change the zero to 9. The 9 may be written above the zero so that it will not be forgotten.

Seventh set of examples: Borrowing across more than one zero.

99	Ones: 12 less 7 is 5 (borrow).
4002	Tens: 9 less 8 is 1.
1587	Hundreds: 9 less 5 is 4.
2415	Thousands: 3 less 1 is 2.

The only explanation to be made is that all the zeros across which we borrow are changed to 9's.

Eighth set of examples: Zeros in the remainder; otherwise mixed as to difficulties. A difficulty occurs in examples in which one or more numbers to the left in the remainder are zeros, but the child readily learns to drop such zeros.

$$\begin{array}{r} 69021 \\ \underline{38212} \\ 30809 \end{array}$$

Under each of these classes of examples enough drill should be given to make the operation automatic before the next class is taken up.

125. Subtraction: second method. — The method described in this section may be called the "taking-away-carrying-method." The same classification of examples may be used here as in the preceding section.

First set of examples: These are solved *exactly* as by the preceding method (see section 124).

Second set of examples: As a preliminary step it should be made clear that adding the same numbers to the subtrahend and minuend does not change the result. This can be done by considering such cases as 14 − 8 = 6, 24 − 18 = 6, etc. We then proceed as follows:

Suppose we are to subtract 56 from 84.

84 14 less 8 is 6 (carry 1).
56 8 less 6 is 2.
28

Since we can not take 6 from 4, we add 10 to the 4, making a sum of 14, and subtract 6 from 14. Since 10 has been added to the minuend we must add 10 to the subtrahend, which we do by adding 1 to the 5. In practice we simply say, "carry 1."

Third set of examples:

738 Ones: 8 less 5 is 3.
465 Tens: 13 less 6 is 7 (carry 1).
273 Hundreds: 7 less 5 is 2.

Fourth set of examples:

4174 Ones: 4 less 2 is 2.
2892 Tens: 17 less 9 is 8 (carry 1).
1282 Hundreds: 11 less 9 is 2 (carry 1).
Thousands: 4 less 3 is 1.

Fifth set of examples:

290	Ones: 10 less 8 is 2 (carry 1).
158	Tens: 9 less 6 is 3.
132	Hundreds: 2 less 1 is 1.

Sixth set of examples:

804	Ones: 14 less 7 is 7 (carry 1).
267	Tens: 10 less 7 is 3 (carry 1).
537	Hundreds: 8 less 3 is 5.

Seventh set of examples:

4002	Ones: 12 less 7 is 5 (carry 1).
1587	Tens: 10 less 9 is 1 (carry 1).
2415	Hundreds: 10 less 6 is 4 (carry 1).
	Thousands: 4 less 2 is 2.

Eighth set of examples:

69021	Ones: 11 less 2 is 9 (carry 1).
38212	Tens: 2 less 2 is 0.
30809	Etc.

The wise teacher will be chary of explanation. By analogy and by imitation the child will mount the successive steps without pain and confusion.

126. Subtraction: third and fourth methods. — The third method may be called the "addition-carrying-method." It differs from the second method in that "adding to the subtrahend" instead of taking away is used. The fourth method is sufficiently characterized by its name — the "addition-borrowing-method." One example will suffice to indicate how these methods may be developed.

	(Third method)		(Fourth method)
4174	Ones: 4 = 2+2.	4174	Ones: 4 = 2+2.
2892	Tens: 17 = 9+8.	2892	Tens: 17 = 9+8 (borrow).
1282	Hundreds: 11 = 9+2.	1282	Hundreds: 10 = 8+2 (borrow).
	Thousands: 4 = 3+1.		Thousands: 3 = 2+1.

The series of examples given in sections 124, 125 may be used in developing either of these methods. The remarks in section 124 apply here also.

127. Comparison of methods of subtraction. — The essential elements to be compared are "taking away" *versus* "adding," and "carrying" *versus* "borrowing." The "taking-away" method has already been compared to the "adding" method (see sections 110–112). The writer believes that for the purpose of the complete process of subtraction the question of "carrying" *versus* "borrowing" is almost as important as that of "taking away" *versus* "adding" and believes it can be proved to a nicety that "borrowing" is more troublesome both to learn and to carry out in practical work. Consider an example, such as:

$$\begin{array}{r}140043\\ \underline{29647}\\ 110396\end{array}$$

By the "carrying" method the numeral 1 is placed (mentally) to the left of each digit in the minuend, whenever this is necessary, in this case giving 13, 14, 10, 10.

This one simple rule covers all cases so far as the minuend is concerned. Whenever 1 is added in this manner in the minuend 1 is "carried" to the next place in the subtrahend. There are no exceptions to these rules and no special difficulties when there are zeros in the minuend.

By the "borrowing" method it is either necessary to write 9's over the zeros or to remember them and also to remember that one has been borrowed from the digit next to the zeros on the left.

If it is concluded that the "adding" method is superior to the "taking-away" method and the "carrying" to the "borrowing," then it follows that the "adding-carrying" method is the best of the four methods considered in sections 124–126. Frequently the so-called "Austrian" method consists of a combination of "adding" and "borrowing." A generation ago the "adding" method was in use in a small minority of our schools but has been gaining very rapidly, especially in the larger cities where the control of such matters is more likely to be in the hands of those who have made a thoroughgoing study of pedagogical questions. The following tables are taken from Jessup and Coffman.[1]

It is not entirely clear which methods are compared. The method named "taking from" in the tables is no doubt the one described in section 124 as the "taking-away-borrowing" method. The so-called "addition" method may be either of methods two or three, described above. Probably the majority used the "addition-borrowing method."

GEOGRAPHICAL DISTRIBUTION OF METHOD IN SUBTRACTION

	ADDITION	TAKING FROM	BOTH	ALL CITIES
North Central	96	134	15	245
North Atlantic	105	107	8	220
Western	24	21	4	49
South Central	16	44	3	63
South Atlantic	11	14	1	26
Counties	12	47	3	82
Totals	264	367	34	685

VARIATION OF SUBTRACTION METHOD IN CITIES

CITIES	ADDITION	TAKING FROM	BOTH	ALL CITIES
1,000,000 and over	2	1	0	3
200,000 to 999,999	7	6	1	14
100,000 to 199,999	7	4	1	12
50,000 to 99,999	8	19	1	28
30,000 to 49,999	13	21	1	35
20,000 to 29,999	22	17	0	39
15,000 to 19,999	9	26	3	38
10,000 to 14,999	32	43	4	79
8,000 to 9,999	28	45	4	77
4,000 to 7,999	124	138	16	278
Totals	252	320	31	603

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[1] *The Supervision of Arithmetic;* The Macmillan Company, 1916, pp. 73–78.

It will be seen that at the time these tables were compiled, the taking-away method was decidedly in the lead in the South Central States and in counties and considerably in the lead in the North Central States. In the other regions — North Atlantic, South Atlantic, and Western—the two methods were used in practically the same number of cities — 140 and 142 respectively.

Of the twenty-nine cities of 100,000 population or greater which reported, sixteen used the Austrian method, eleven the "taking-away" method and two used both. Of the one hundred thirty-one cities of 20,000 population over fifty-nine used the Austrian method, sixty-eight the "taking-away" method and four both. In view of the larger number of large cities using the Austrian method it seems safe to conclude that at least as many children were at this time using the Austrian method as the other method. (It is a pity the authors do not tell us which one of the three cities, New York, Chicago, and Philadelphia, was using the "taking-away" method.) It is probable, however, that among the small towns and villages not reporting and in the rural schools the older method was decidedly in the preponderance.

128. Multiplication by one figure numbers. — The process of multiplication is sometimes defined as a short process of adding in which the addends are all the same number. Whatever criticism this definition may be subject to, it is a fact that less than one thousand years ago a number like 122 was multiplied by 4 by adding. In teaching multiplication by a one figure multiplier, such as 4, it should be shown that the same result is obtained whether the multiplicand is used as addend four times or multiplied by 4. By using this method of approach the process of multiplication can be readily understood. Carrying, for example, is thus shown to depend on the same principle as in addition. There should be considerable practice on this kind of examples before larger numbers are

$$\begin{array}{r}122\\122\\122\\122\\\hline 488\end{array}$$

used as multipliers. For one thing such practice will serve to strengthen the child's hold on the multiplication table. Indeed there is no reason why this method of multiplication should not be begun as soon as some of the tables are known and the corresponding problems in addition have been mastered. That is, 485 may be multiplied by 4 if the table of 4's is known.

This would seem an opportune point at which to state an important principle of teaching:

Every new process or principle should be analyzed so as to exhibit separately all the elements which have already been studied.

By so doing several distinct ends are achieved: (1) the best possible review is given of parts of what has gone before; (2) by isolating the new elements they appear less formidable and are mastered much more easily than when mixed in a chaos with much that has already been learned — a chaos which is likely to appear totally strange; (3) finally may we not hope that persistence in this sort of analysis, performed by the textbook and the teacher in the early stages but gradually taken over by the pupil himself, will at the end of eight or twelve or sixteen years of school life have a tendency to make a person more analytical in his thought? May not this have some effect in decreasing the number of people who rashly divide the world into two parts — that which is "perfectly lovely" and that which is "perfectly horrid"?

129. Basic ideas in multiplication. — The process of multiplication when either multiplier or multiplicand or both are greater than 9 is based on a series of fundamental principles. While these principles should be pointed out to the pupils sparingly and only to the extent that they are capable of understanding and making use of them (see section 135), they should be perfectly familiar to the teacher. These principles are:

1. To multiply the sum of two numbers, multiply each one separately and add the products.

Thus, in multiplying 46 by 7, we really multiply 6 and 40 by 7 in succession and add the products.

2. To multiply by the sum of two or more numbers, multiply by each number separately and add the products.

Thus, to multiply a number by 34, we multiply by 4 and then by 30 and add the products.

Indeed multiplying 67 by 34 involves the same principles as finding the product, $(a + b)(c + d) = ac + ad + bc + bd$ in algebra. This is shown in the operation carried out in detail at the right. In practice we have contrived to multiply 67 $(= 60 + 7)$ by 4 and 67 by 30, so as to have each product appear as one number.

$$\begin{array}{rr} & 67 \\ & \underline{34} \\ 4 \times 7 = & 28 \\ 4 \times 60 = & 240 \\ 30 \times 7 = & 210 \\ 30 \times 60 = & \underline{1800} \\ & 2278 \end{array}$$

3. To multiply by a product of two numbers, multiply by one of the factors and then multiply the product by the other factor.

Thus, to multiply by 30, multiply by 3 and the product by 10.

4. To multiply by 10, annex a zero to the right or, what amounts to the same thing, move each figure in the number one place to the left. To multiply by 100, annex two zeros or move each figure two places to the left and so on.

Oral drill may be given in multiplying by such numbers as 20, 30, 40, . . . 200, 300, etc. Products of such combinations as 20 × 4, 60 × 8, 300 × 6 should be read rapidly at sight.

The child may then be led to understand that to multiply by 16, he should multiply by 6 and 10 and add; to multiply by 47, he should multiply by 7 and 40 and add.

130. Multiplication. — To multiply by a number such as 347 (shown on page 237), we multiply by 7, 40, and 300 and add the products. The zeros may be written down at first for the purpose of explanation. It will be clear enough later that they may be dropped at any time provided the other figures are placed in their proper places. The teacher should bear in mind that, especially in the beginning, all attempts to rationalize processes by any but the most simple explanations are often apt to bring serious misgivings. Overexplaining may be worse than no explaining (see section 135).[1]

$$\begin{array}{r} 814 \\ \underline{347} \\ 5698 \\ 32560 \\ \underline{244200} \\ 282458 \end{array}$$

To multiply by a number like 208, the work may first be put down as shown at the right, dropping the zeros after a few examples have been worked.

$$\begin{array}{r} 396 \\ \underline{208} \\ 3168 \\ \underline{79200} \\ 82368 \end{array}$$

At some time, possibly later in the course in grade arithmetic, it should be pointed out that the best way to multiply two numbers, such as 15200 and 430, is to multiply 152 by 43 and annex three zeros. The work is carried out exactly as if there were no zeros and then the proper number of zeros annexed.

$$\begin{array}{r} 15200 \\ \underline{430} \\ 456 \\ \underline{602} \\ 6476000 \end{array}$$

131. The multiplier and the multiplicand. — If frequent repetition (by word of mouth and in print) insures permanent lodgment in the memory, there should be small danger of forgetting that the multiplier is always an abstract number while the multiplicand may be either abstract or concrete, and that the product is the same kind of number as the multiplicand. Quite true! Quite true! But truth, what monstrosities are said to be of thy begetting! We are told, for instance, that to find the daily pay roll of 297 men at $4.00 a day we must multiply 4 by 297 and not 297 by 4. No sane person outside an educa-

$$\begin{array}{r} 4 \\ \underline{297} \\ 28 \\ 36 \\ \underline{8} \\ 1188 \end{array} \qquad \begin{array}{r} 297 \\ \underline{4} \\ 1188 \end{array}$$

---

[1] "At any rate the fact is sure that most pupils do not learn the multiplications by deductive reasoning or understand them as necessary consequences of abstract principles." Thorndike, *op. cit.* (p. 42), p. 67.

tional museum would ever choose the first of these methods. In section 194 a fuller discussion is given of abstract and concrete numbers. It will suffice here to lay down the general rule that to find the product of two numbers that one should usually be used as the multiplier which contains the smaller number of figures not counting zeros. Thus to find the product of 198 and 12, use 12 as the multiplier; to find the product of 2600 and 491, use 2600 as the multiplier; and to find the product of 1090 and 126, use 1090 as the multiplier. This rule is the one ultimately to be used; the only question is as to the point at which the multiplication by 2600 by the last method described in section 130 should be introduced. In practical work the concrete or abstract quality of the numbers should have nothing whatever to do with the question as to which should be the multiplier and which the multiplicand. The character of the product, on which so much stress has been laid in discussions of this topic, must in any event be decided upon in some other way. We have deceived ourselves beautifully.

132. Division: one-figure quotients and one-figure divisors. — All possible cases of division where both divisor and quotient are 10 or less and where there are no remainders are learned as number facts related to the multiplication table. That is, $56 \div 8 = 7$ and $56 \div 7 = 8$, *because* $7 \times 8 = 56$ (see section 113). These should be studied so fully along with the multiplication table that they can be given without hesitation. Indeed, when the idea of the relation of multiplication to division is once clear, these facts will be known just as well as the multiplication table (and no better).

A good preparation for further work in division consists in examples like finding the quotient and remainder in $45 \div 6$, etc. Rapid oral drills should be given on examples of this sort until the results are stated accurately and without hesitation. This also affords valuable drill in the multiplication table since the facts of the table are now used in *combination* with other processes.

133. Short division. — Little or no effort should be made to rationalize the process of division beyond the type of examples considered in the last section. So-called "long" and "short" division are *exactly* alike in so far as the process goes. They differ essentially in this, that in long division each step is reduced to writing while in short division much of the work must be carried in the memory. Aside from the difficulty of deciding on each quotient figure in case the divisor is large the whole trouble encountered in division lies in the sequence of steps to be taken in carrying it out. This sequence of steps being the same in both long and short division, it may be learned most easily by using small divisors such as are usually used in short division. Moreover, it is well at the outset to leave the mind as free as possible by writing down the numbers exactly as in long division.[1] Hence this process should be studied exactly as in long division. Consider the example in the margin.

$$\begin{array}{r}
65812 - 4 \\
6\,\overline{)\,394876} \\
\underline{36} \\
34 \\
\underline{30} \\
48 \\
\underline{48} \\
7 \\
\underline{6} \\
16 \\
\underline{12} \\
4
\end{array}$$

Starting from the left end of the dividend we take enough of it so that 6 is contained at least once but not ten times. That is, we take 39. 6 is contained 6 times in 39 with a remainder of 3. This we find by subtracting $6 \times 6 = 36$ from 39. Then bring down the next figure, 4, from the dividend, and divide 34 by 6, and so on. Care should be taken to place the 6, the first figure of the quotient, directly above the 9 in the dividend.

---

[1] The introduction of the long division form by means of divisors small enough for short division is violently opposed by some. It is said that children who use the long division form for such examples will never use short division. This argument would seem to be based on the assumption that the child does not know that the "long" and "short" division processes are exactly alike and that in short division the "form" written down is abbreviated by carrying more of the work in the memory. If such ignorance is necessary to get children to use short division, it may be questionable whether short division is worth while. The writer knows from his own experience that the difficulty of changing from the "long" to the "short" division form is very largely imaginary.

After the child has had a certain amount of practice on this kind of examples, he should be given one in which a zero appears in the quotient. That is, examples like the one shown here should be introduced but care should be taken not to introduce this kind *before* the type given above is mastered. "One difficulty at a time" is the slogan. By the time examples of this second type have been mastered *and not until then* should the abbreviated form of record be used. It is then shown how easily the quotient and remainder may be written down directly. The child says: 39 = 6 × 6 with 3 over. The 6 is written down as before and *3 retained in memory* instead of written down. The 3 with the 4 following makes 34. 34 = 5 × 6 with 4 over, and so on to the end of the problem. Now go over the same exercises that were solved by the method given above. The child will like it. He has learned to shorten the work and feels fine about it. The writer is, of course, keenly aware of the opposition in many quarters to this procedure and equally well aware of the futility of arguing against it. He can only hope that this book may survive the indiscretion.

$$\begin{array}{rl}
 & 4203 - 3 \\
7 & \overline{)\,29424} \\
 & \;\;\underline{28} \\
 & \;\;14 \\
 & \;\;\underline{14} \\
 & \;\;24 \\
 & \;\;\underline{21} \\
 & \;\;3
\end{array}$$

$$\begin{array}{rl}
 & 65812 - 4 \\
6 & \overline{)\,394876}
\end{array}$$

134. Long division. — In order to teach long division (perhaps the most trying subject in the whole arithmetic when the age of the child is considered) properly, the teacher must have clearly in mind at the outset the successive steps which make up the process. These steps may be enumerated as follows:

1. Find the first partial dividend.
2. Estimate the first figure in the quotient.
3. Multiply the divisor by the estimated quotient figure.
4. Compare the product with the first partial dividend.

5. Subtract the product from the partial dividend.

6. Bring down one figure from the dividend to form the next partial dividend and repeat the first five steps.

These steps are already partly familiar from short division. We now focus the attention directly upon these for the purpose of studying them more particularly.

An example, having as small a number as 12 for a divisor, may be used as a text as these steps are explained. It should be noted, of course, that these steps are precisely those used in short division.

When the sequence of steps is understood clearly, each step should be studied separately. Some of them are difficult and will require much thought and drill while others are easy and will require but passing comment.

Step 1 will cause little trouble. Suppose the divisor is 281 and the dividend is 124768. It is clear at once that 124 is too small for the first partial dividend since it does not contain the divisor once. 1247 contains the divisor; and since it is the smallest number of figures taken from this number which does contain it, this is the first partial dividend. This is a case where the child should not be required to give reasons. Lately I asked a boy who could pick out the first partial dividends unerringly to tell how he knew. He said "I know all right, but it is too long a story to tell." He was right. Many a girl can dance divinely though she can not tell how she does it.

Step 2 is by all odds the most difficult and should be given separate attention until it can be taken fairly rapidly and with a high degree of accuracy. While it is true that subsequent steps constitute a check on this one, it is also true that in practical work a wrong quotient figure causes enough trouble to make it really serious. If the finished work is to look at all neat, the whole example must be repeated. In order to develop reasonable skill in finding quotient figures, drill must be given in estimating the quotients in such examples as $491\overline{)3934}$. The difficulty in estimating quotient figures depends largely upon the character of the divisors, and these should be introduced in order of difficulty. The easiest divisors are of the types 19, 21, 29, 31, etc., since they are close to the numbers 20, 30, etc. Such numbers as 28, 32, 38, 42, etc. are nearly as easy, while 24, 26, 34, 47, 56, etc. are much more difficult. Among three-figure numbers the easier ones are those that are near multiples of 100, such as 296, 608, 798, 813.

While in general 491 is an easy divisor, $491\overline{)3934}$ presents a difficulty. If we use 500 as an approximate divisor, the quotient is 7, while as a matter of fact the quotient in the proposed problem is 8. In using approximate divisors allowance must be made because they are always a little too large or too small. This is an inescapable difficulty and must be faced squarely. However, care and experience — in this case much experience — will gradually diminish the errors in estimated quotient figures.

The point urged here is that much drill on this step should be given, and for the purpose of this drill the step should be entirely isolated from the rest of the division process. The following device may serve the purpose: Write, say ten examples in division on the board. Be sure that in each case the quotient is a one digit number. Let the children estimate the ten quotients rapidly and write them down. Then let them exchange papers, make sure of the proper quotients, and mark the errors. Frequent drills of this sort should be had until the children become expert in estimating quotient figures *correctly*. Since this step contains more than half the trouble in long division and since long division is the backbone of at least one semester's work in arithmetic, the attention to it which is recommended here is surely not too great. One of the weakest spots in our teaching of arithmetic is perhaps our failure to isolate this troublesome step for separate and drastic treatment.

The remaining steps in the process are simple and will not require much separate attention. Such minor matters as zeros in the quotient should, of course, receive special attention.

In such examples as $4800\,\overline{)863000}$ two zeros should be stricken out of both dividend and divisor, reducing the example to $48\,\overline{)8630}$.[1]

The teaching of division may therefore be outlined as follows:

1. Drill in giving quotients and remainders in such combinations as $46 \div 7$.
2. Study of the sequence of steps in division by using small divisors, such as 6 or 9, and writing out the full form as in long division.
3. Continue the process under 2, using larger easy divisors, such as 21, 31, 19.
4. Practice in estimating the quotients in such examples as: $29\,\overline{)187}$, $42\,\overline{)364}$, $88\,\overline{)659}$, $212\,\overline{)1764}$, $593\,\overline{)4687}$, $814\,\overline{)4916}$.
5. With the practice described under 4, intersperse regular long division, using such divisors as have already been mastered.
6. Attend to such special difficulties as zeros in the quotient.

There are two essential difficulties to be mastered: (a) the sequence of steps comprising the whole process, and (b) the estimating of the quotient figures.

135. Rationalizing the processes. — The general question of rationalizing the processes has been considered incidentally in various sections. However, it would seem to merit more direct treatment. What is meant by "rationalizing a process"? When speaking of a mathematical process, it is customary to regard the rationalizing of it as consisting of the following steps:

1. A formal definition of the process.
2. A *deductive* proof that a certain set of steps actually does constitute such a process.

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[1] For details of this method, see Lennes and Jenkins, *op. cit.* (p. 150), Book I, pp. 239, 253.

The whole matter is indeed much more complicated than would seem to be indicated by these statements. We are on the edge of the whole subject of the foundations of mathematics. We may, however, modify the requirements by saying that by rationalizing the process we mean simply that we satisfy the mind *somehow*, no matter how, that the process *must* lead to the correct results. Even this is far too stiff a program for the elementary grades. Much of what we do when we try to rationalize a process in arithmetic is to show that it "sort of goes with" certain other processes or facts that we hold to be true. We are seeking simply to show a similarity between the process and what is already familiar to the child. When this is not expedient, we do not even attempt this much as in the case of long (and also short) division.

There is one method of comparing the new process with what is already known, and this may be used almost universally and is very convincing to the child: We may show that the results obtained will pass certain tests. Thus we may test the result of division by multiplying the quotient by the divisor and adding the remainder and comparing the result with the dividend. But the child does not know that the remainder added to the product of divisor and quotient will equal the dividend! True enough, but we can tell him!

Another method by which the processes of arithmetic may be made clear to young children is simple induction. We may show that a process leads to results known to be correct when the numbers are very simple. Thus, in long division, we may carry out the complete process, using $3\,\overline{)\,77}$. The result is 25 with a remainder of 2 — a result which the child can easily verify. After working a few such simple examples and verifying the results, the child is perfectly willing to believe that the process *must* lead to correct results in all cases. This method is of very general application and is one of the most effective means for making the processes of arithmetic look reasonable to the child.

But when all is said and done, "it is a very wise teacher who knows when the time has or has not arrived for a more thoroughgoing explanation."

136. Practical needs of the fundamental processes. — It is not uncommon these days to hear people say that adding, subtracting, multiplying, and dividing are now obsolete arts (might they also say lost arts?); that the computing machine does all these and does them more rapidly and more accurately than a human being could ever do them. But is this true in general practice? In a town of 15,000 how many concerns do their bookkeeping on a machine? Usually a small percentage will be found even to possess the simplest adding machine. But it is not in business offices only that these operations are of importance. It is now fully understood that the best type of housekeeping is impossible unless a complete account of expenditures is kept. The writer knows personally many cases where this is not done simply because it is too tedious to add the various columns each month. A common remark is, "I could never get the accounts to come out straight and so I quit it." Reasonable facility in performing the four operations of arithmetic would enable these people to go on with their household accounts without any sense of hardship, and in many cases this would make possible savings which would prevent untold (and concealed) agony. No one can spend money wisely unless he knows how he is spending it, and no one can know this for any length of time unless he keeps accounts. This is only one of the many relations in life where facility and accuracy in handling numbers is of grave importance. There are few subjects indeed in the whole school curriculum that are of greater practical importance than this.[1]

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[1] "If social demands are to be taken as a criterion for judging the value of the various subjects, then the school should vigorously insist upon a mastery of the materials and proc-

137. Drills in fundamentals. — Two points involved in the question of drills in fundamentals have been under discussion. One of these points relates to the fundamental question as to whether separate drills should be given at all or whether the pupil should obtain all his drill by performing the fundamental operations as they occur in solving concrete problems. The other point relates to the manner of giving the drills in case it is granted that they should be given at all.

On the first of these points we quote Starch[1] at length:

> A considerable number of careful experimental studies on the influence of drill are now available and without exception they show drill to be distinctly valuable. Thorndike had nineteen university students practice adding 48 ten digit columns of figures daily for seven days. While the work required on the average less than an hour in all, there was an improvement of 29.0% over the original rate.
>
> J. C. Brown performed two elaborate comparative experiments to determine whether children under controlled school conditions profit more by giving a small part of each class period to drill or by spending the entire

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esses of arithmetic. This subject so far as it relates to common community life increases in serviceableness in proportion to the degree to which its fundamental processes have been reduced to the plane of habit." Brown and Coffman, *op. cit.* (p. 31), p. 92.

"The use of arithmetic in our daily lives is of greater importance than is generally understood. No individual can regulate his expenditures intelligently unless he keeps an account showing his income and the purposes for which he expends it. The expenditures of a family can never be wisely made unless a complete household account is kept, running over a series of years. No woman can keep house as economically as possible unless she figures the value of the food she buys at varying prices.

"A farmer, no matter how intelligent and successful, will be more successful if he keeps accounts showing which cow or which field pays him the best. A few minutes spent each day in intelligent application of the arithmetic learned in the grade schools will pay a higher income per hour than any other work we are likely to do. . . .

"Experience has shown that people will avoid doing whatever is difficult and laborious. Only those persons who can perform the operations of arithmetic with ease, accuracy, and comfort will use them freely. If you can add or multiply only with difficulty and with a feeling of dislike and fatigue, the chances are that you will do no more of it than seems absolutely necessary. The result will be that you will neglect to make that use of arithmetic which is necessary for the highest success. Hence it is of the greatest importance that you should practice on the fundamental operations until you can perform them rapidly and with practically absolute accuracy." Lennes and Jenkins, *op. cit.* (p. 150), Book III, p. 258.

[1] *Op. cit.* (p. 3), pp. 398–400.

period in ordinary routine work in arithmetic. . . . Drill on the four fundamental operations was given to one half of the group for the first five minutes of each class period of twenty-five minutes. About half the drill was oral and half was written. The drill lasted thirty periods. In the second experiment 222 children were used and the drill was given for twenty periods. . . .

In both the experiments there was a decided advantage in using a part of the recitation period for drill. In the first experiment the drilled group gained about twice as much as the undrilled group; while in the second experiment the drilled group improved about two and one half times as much as the undrilled group. The sixth grade gained the most (35%) and the eighth the least (13.8%). In order to determine whether group I gained on fundamentals at the expense of reasoning, both groups were tested in arithmetical reasoning before and after the drill. Here again the drilled group did better, making a gain of 6.3%, while the undrilled group gained only 3.0%. . . .

Since the improvement in reasoning, which had not been drilled at all, was almost exactly the same proportionately as the processes which were drilled, it suggests that the drill had a tonic effect upon the remainder of each recitation period following the drill, to which much of its value was due.

In order to discover the permanency of the effects of drill, Brown tested both groups once more after a twelve-weeks vacation and found that the drilled group was also superior in retention, having lost .2% while the undrilled group had lost 2.29%.

The same experiment was repeated by F. M. Phillips. He had 69 children for subjects and gave to one group drill in fundamentals and in reasoning, both oral and written, for eight weeks. Neither teachers nor students knew the purpose of the tests. He found that "the improvement in fundamentals of the combined drill groups was 15% greater than that of the nondrill groups. In reasoning the drill groups improved 50% more than the nondrill groups. . . . The greatest gains were made in the sixth grade and the least in the eighth." Almost all the gain on fundamentals was in multiplication.

On the second point there seems to be less definite evidence. There undoubtedly is, however, a general drift toward the practice of giving short drills at increasing time intervals throughout the grade-school course. Such practice is no doubt founded on sound principles and may safely be continued unless further investigation should furnish entirely new light on this question.

REVIEW QUESTIONS

1. Explain how addition by endings is related to column addition. Should "carrying" be used in addition by endings? Is carrying possible in single column addition? Explain how addition by endings is best developed. Should it ever be taught after carrying has been learned? Why? (§§ 120, 121.)

2. How should carrying in addition be taught? Discuss the danger of overexplaining. (§§ 122, 135.)

3. Discuss the early introduction of large numbers. What is commonly meant by "comprehending" large numbers. (§ 123.)

4. Describe the four methods of subtraction considered in sections 124 to 127. Compare these with respect to brevity, ease of learning, and especially in regard to the comparative difficulties involved in "borrowing" and "carrying."

5. Give the basic ideas involved in multiplication and the successive steps in the order in which they should be introduced in building up this process. Note again the danger of overexplaining. (§§ 128–130.)

6. Give the types of examples in division in the order in which they should be introduced. (§§ 132–134.)

7. What are the successive steps in long division? Which of these steps is the most difficult, and how may separate practice on this step be given? (§ 134.)

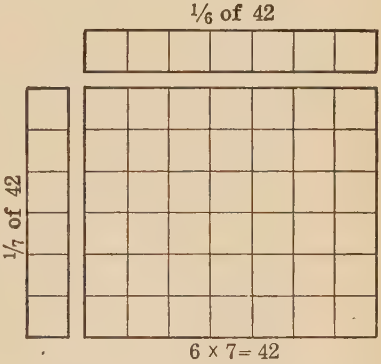
8. Discuss rationalizing of the processes. What means of rationalization may be used? (§ 135.)

9. Comment on the statement: "The fundamental processes are no longer important since they are now carried out on the computing machine." What can be said about the present practical needs of skill in these processes? (§§ 136, 137.)

CHAPTER VIII

COMMON FRACTIONS

138. Early introduction of fractions. — We have already considered the introduction of fractions in connection with the multiplication table. Suppose we are studying 6 × 7 = 42. From the figure the meaning of 1/6 of 42 and 1/7 of 42 is made clear; and it is also made clear that 1/6 of 42 = 7 and 1/7 of 42 = 6, *because* 6 × 7 = 42. In this manner we get a fairly good idea of what are called unit fractions. Other fractions may now be introduced. If 1/7 of 42 = 6, then the child has no difficulty in seeing that 2/7 of 42 = 12, 3/7 of 42 = 18, etc. This is a simple and natural extension of the study of fractions and affords a good review of the multiplication table.



Fractional parts of units are also studied in the second, third, and fourth grades. The child learns very early to know halves, thirds, and fourths. The addition and subtraction of like fractions (fractions having the same denominator) is of course easy and usually receives some consideration in the third grade and certainly in the fourth. Aside from these incidental and informal considerations, fractions are usually deferred until the fifth grade.

139. Fractions in common use. — When the study of fractions is taken up formally, a systematic effort should be made to learn what fractions are in common use. Catalogues of the various special lines of goods are the best sources of information. Standard commercial sizes of the paper of notebooks and of cards, of negatives and photo prints; the widths of ribbons; the lengths of screws, nails, and bolts; the diameters of bolts and nuts, and particularly the standard diameters of wire; the thickness of steel plate; the diameters of shot and bullets; the dimensions of parts of instruments and machinery — all are expressed in terms of fractions which include nearly all that are in practical use. In many cases the children can bring catalogues containing such information. In other cases the teachers will have to help. The simplicity of the fractions in common use will be a revelation both to pupil and teacher — perhaps not least to the teacher. Such an investigation was a decided revelation to the writer of this book. Fractions, such as 3/7 or 5/9, are practically never used. Fractions which have denominators other than 2, 4, 8, 16, 32, 64 are used but seldom. It will be surprising to find how seldom even such fractions as 1/3 and 2/3 are used.

The lists of fractions given below are indicative of what will be found in such a quest. For a number of reasons it is better to compile such lists in the schoolroom from material gathered in the neighborhood than to take them out of a book.

Paper is cut into sizes given in inches and fractions of an inch. The fractions used in describing school blank books are: 1/2, 1/4, 3/4, 1/8, 5/8, and 7/8. In describing commercial sizes of stationery 1/2, 1/4, 1/3, 2/3, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, 1/16, 3/16, 7/16, 9/16, 11/16, and 13/16 are used. In giving sizes of negatives and lantern slides 1/2, 1/4, 3/4, 7/8, and 5/16 are used. In describing commercial sizes of cards for mounting photographs, 1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, and 9/16 are used.

The fractions used in measuring the thickness of wires under 1/2 inch are: 1/4, 1/8, 3/8, 1/16, 3/16, 5/16, 7/16, 1/20, 1/32, 3/32, 5/32, 7/32, 9/32, 11/32, 13/32, 15/32, 1/40, 1/64, 3/64, 5/64, 7/64, 9/64, 11/64, 13/64, 15/64, 17/64, 19/64, 21/64, 23/64, 25/64, 27/64, 31/64, 1/80, 3/80, 9/128, 1/160, 3/160, 7/160, 9/160, 3/320, 7/320, 9/320, 11/320, 5/640, 7/640, 9/640, 11/640, 9/1280, 11/1280, 13/1280, 17/2560.

Commercial lengths of wire nails less than one inch are 5/6 inch and 7/8 inch. From one inch they increase by 1/4 of an inch to 3 1/2 inches and then by 1/2 of an inch to 6 inches.

Diameters of washers for bolts, given in inches are: 3/16, 3/8, 7/16, 5/8, 11/16, 13/16, 31/32, 1 1/8, 1 3/8, 1 1/4, 1 1/2.

Diameters in inches of standard shots are:

Fine dust	3/100	No. 7	10/100	B	17/100
dust	4/100	No. 6	11/100	BB	18/100
No. 12	5/100	No. 5	12/100	BBB	19/100
No. 11	6/100	No. 4	13/100	T	20/100
No. 10	7/100	No. 3	14/100	TT	21/100
No. 9	8/100	No. 2	15/100	F	22/100
No. 8	9/100	No. 1	16/100	FF	23/100

Buck Shot:

No. 3	25/100	No. 0	32/100	Balls	38/100
No. 2	27/100	No. 00	34/100	Balls	44/100
No. 1	30/100	No. 000	36/100		

The only numbers used as denominators in the fractions given above are 2, 3, 4, 6, 8, 16, 20, 32, 40, 64, 80, 100, 128, 160, 320, 640, 1280, 2560.

140. Fundamental property of fractions. — The fundamental property of a fraction is the following:

The numerator and denominator of a fraction may be multiplied or divided by the same number without changing its value.

This is the principle used in reducing a fraction to its lowest terms and in reducing fractions to a common denominator. The same principle is also used in ordinary cancellation. In a slightly modified form it is used to simplify problems in long division (see section 134), in division of decimals (see section

162), and in division of mixed numbers (see section 147). The modified form used in these applications is:

The dividend and divisor may be multiplied or divided by the same number without changing the value of the quotient.

Care should be taken to connect all these applications and to make it clear that they depend upon the same principle. Ways of doing this are indicated in the sections cited above.

At the point where this principle is introduced, pains should be taken to make it perfectly clear. Remember that the simplest and most familiar illustrations are the best. Thus a half pie is equal to two quarters of a pie. That is, 1/2 = 2/4. By means of diagrams, such as are found in any good textbook, it should now be made clear that 1/2 = 3/6, 1/3 = 2/6, 2/3 = 4/6, 1/4 = 2/8, 3/4 = 6/8. When each of these and possibly a few more have been thoroughly illustrated, the child is ready to understand the meaning of the general principle and to believe it. At first it may be well to state it in two parts, viz.:

1. *The terms of a fraction may be multiplied by the same number without changing its value.*
2. *The terms of a fraction may be divided by the same number without changing its value.*

The first statement comes directly from the equalities written above. The second statement comes from the same equalities when read backward. Or they may be rewritten: 2/4 = 1/2 3/6 = 1/2, etc. Then the teacher says: "Let us see whether we can state both of these principles in one sentence," and the result is the statement given at the head of this section. The importance of reducing such a principle to definite form for reference and use has been discussed in Chapter IV (see section 99).

141. Cancellation. — The subject of cancellation should *not* be introduced before the principles of section 140 have been studied. It does not matter how soon after that it is introduced. In ordinary cancellation we are really dealing with an indicated quotient or division. That is, one number is to be divided by another. At this stage it should therefore be pointed out that a fraction may be regarded as an indicated division. That is, 6/2 may be regarded as 6 divided by 2, and 7/8 may be regarded as 7 divided by 8. There is no need for elaboration or attempted proof of this fact. It is better not to say much about it. The child will have no trouble with this idea if the teacher or the textbook does not make the trouble for him.

There is another difficulty in cancellation which should receive attention. Suppose the child knows that the same number may be cancelled in both numerator and denominator. What will he do with the expression, $\frac{18}{4 \times 8 \times 16}$? Will he write it $\frac{\overset{9}{\cancel{18}}}{4 \times 8 \times \underset{8}{\cancel{16}}}$ or $\frac{\overset{9}{\cancel{18}}}{\underset{2}{\cancel{4}} \times \underset{4}{\cancel{8}} \times \underset{8}{\cancel{16}}}$? The writer has seen all thinkable possibilities tried and that by first year students in a high school. A thoroughgoing treatment would no doubt be premature here. The statement "cancel each factor once in the numerator and once in the denominator" will help. Examples showing how it is done will assist the student, for he works by analogy. However, all these devices will fail when it comes to an expression like $\frac{2 \times 12}{3 \times 8 + 4 \times 6}$. But it will probably be best to avoid this type of examples. Of course, the strongest pupils could learn to handle such expressions effectively, but in practice they receive but little real instruction. Their school life is one long holiday, bespattered with intervals of *ennui* and tedium. There is nothing for them to do except small chores.

142. Use of figures in teaching fractions. — Figures are used not only to show the meaning of fractions but also to illustrate and verify many of the processes.[1] The opposite page taken from Lennes and Jenkins' Applied Arithmetics, Book II, page 64, indicates what the writer has in mind:

The purpose of this page is not to lead the child to discover for himself the rule for finding the product of two fractions, but to provide material which will give meaning to the rule and which will make it seem plausible when it is once given to him by the teacher or read in the text. The old re-discovery program must be abandoned — in fact it was never taken seriously in a practical sense. Life is too short; real discoveries are rare, and even the greatest make but few in a lifetime. What can a child do?

We can, however, give body and meaning to the rules in such manner that the child feels he is talking about something real and not merely using words. This is largely the purpose of using pictures and concrete objects to illustrate the operations on fractions.

143. Reducing fractions to common denominators. — The complicated rules for reducing fractions to common denominators are rarely needed in practice. Fractions used in practical affairs are so simple that the operations required on them may largely be performed mentally. The steps involved are the following:

1. Find the L. C. M. of the denominators.
2. Find by what number the terms of each fraction must be multiplied to reduce it to the required denominator and multiply both terms by this number.

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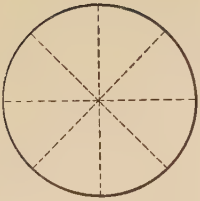
[1] The writer does not share the objection urged in some quarters against the use of objects the exact quantity of which has not been determined.

"It is a great pity that our authorities use these unmeasured units so much, particularly in fractions. Half an apple, half a pie, is a practical, not a mathematical expression at all. To make it mathematical we should have to know just how great the whole is — how many ounces or cubic inches." McLellan and Dewey, *op. cit.* (p. 31), p. 49.

Whether half an apple and half a pie are mathematical expressions or not, they serve admirably to give real meaning to the fractions that the child must use.

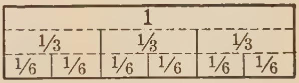


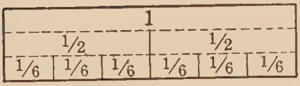
1. Point out 1/2 of the circle, 1/4 of it, 2/4, 3/4, 4/4, 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 8/8.

2. Read and supply the missing numbers in the following and find each result in the circle: 1/2 of 1/2 = ?, 1/2 of 1/4 = ?, 1/2 of 3/4 = ?

3. In the next figure point out 1/3, 2/3, 1/6, 2/6, 3/6, 4/6, 5/6, 6/6.

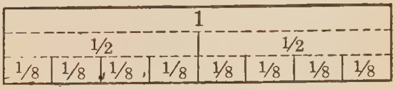
4. Supply the missing numbers in 1/2 of 1/3 = ? and find the result in the figure.

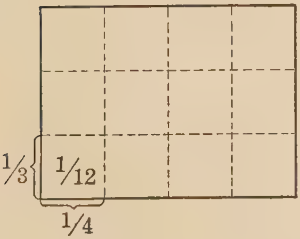




5. Supply the missing numbers in the following and find each result in the figure: 1/3 of 1/2 = ?, 2/3 of 1/2 = ? Compare 1/3 of 1/2 and 1/2 of 1/3.

6. Supply the missing numbers in the following and find each result in the figure: 1/4 of 1/2 = ?, 3/4 of 1/2 = ?





7. Show 1/3 of this rectangle. Also show 1/4 of the rectangle. Show 1/3 of 1/4. What fraction of the whole rectangle is this? Show 1/4 of 1/3. What fraction of the rectangle is this? Supply the missing numbers in 1/3 of 1/4 = ?, 1/4 of 1/3 = ?

Courtesy of J. B. Lippincott Co.

The usual form serves rather effectively to conceal the essential nature of the process. The writer reduced hundreds of sets of fractions in his school days without suspecting in the slightest that the two steps given above were involved; and it may not be too severe a reflection on the schools of twenty odd years ago if he confesses that he taught arithmetic in a rural school while equally innocent and unsuspicious. Of course, things are different now.

As has been indicated above, any set of fractions which are at all likely to occur in practical affairs may be reduced to a common denominator without any special devices and the results put down at once. Suppose it is required to reduce $1/2$, $3/4$, and $2/3$ to equivalent fractions having a common denominator. It is seen at a glance that 12 is the L. C. D. Then the child can write down at once $1/2 = 6/12$, $3/4 = 9/12$, $2/3 = 8/12$. If we care to do so, we may see in the solution of this example the steps involved in the formal process used for complicated fractions.

144. The "carrying" method of subtracting fractions. — The processes of adding and subtracting fractions when they have been reduced to a common denominator are obvious and need no comment. It is frequently said, however, that the "carrying" method in subtraction can not easily be extended to the subtraction of mixed numbers. But this statement is not true as is shown by an example: Let it be required to subtract $4\frac{5}{8}$ from $12\frac{3}{8}$. This is usually done as follows:

$$\begin{array}{rcl} 12\frac{3}{8} & = & 11^{11}/_{8} \\ \underline{4\frac{5}{8}} & = & \underline{4\frac{5}{8}} \\ & & 7^{6}/_{8} \text{ or } 7\frac{3}{4} \end{array}$$

It may, however, be done equally easily as follows:

$$\begin{array}{rl} 12\frac{3}{8} & \quad \text{Fractions: } {}^{11}/_{8} = \frac{5}{8} + {}^{6}/_{8} \text{ or } \frac{3}{4} \text{ (carry 1)} \\ \underline{4\frac{5}{8}} & \quad \text{Ones: } 12 = 5 + 7 \\ 7\frac{3}{4} & \end{array}$$

Instead of reducing one of the 12 units to 8ths, 1, or 8 /8, is added to the minuend and then carried to the subtrahend as in the subtraction of integers. The second method is very little if any more difficult than the first and the child who has learned the "carrying" method of subtraction in the earlier grades should continue it here.

145. Multiplication of fractions. — The use of figures in illustrating and making plausible the process of multiplying fractions has already been considered in section 142. It remains to point out the sequence of introducing the difficulties. The following types should be introduced in order.

1. $3 \times \frac{5}{8}$
2. $\frac{5}{8} \times 3$
3. $\frac{1}{3} \times \frac{1}{4}$
4. $\frac{1}{3} \times \frac{5}{8}$
5. $\frac{2}{3} \times \frac{1}{5}$
6. $\frac{2}{3} \times \frac{5}{8}$
7. $5 \times 2\frac{2}{3}$
8. $4\frac{3}{4} \times 5$
9. $3\frac{1}{2} \times 2\frac{3}{4}$

Example 1 is obvious: $3 \times \frac{5}{8} = \frac{5}{8} + \frac{5}{8} + \frac{5}{8} = \frac{15}{8} = 1\frac{7}{8}$. At this point let the child try to formulate a rule. Finally he should get it into some such form as: *To multiply a fraction by an integer, multiply the numerator by that integer.*

This rule should not be formulated for the purpose of remembering it. By the time the child has formulated the rule he will understand the subject matter covered by it sufficiently well so that he will not need the rule. In any event it is good drill in English composition (see section 98).

Example 2 may be solved at once when we remember that to find the product of two numbers either one may be used as the multiplier. Constant use was made of this principle when learning the multiplication table. It (the commutative law of factors) is now recalled and hence $5/8 \times 3$ is the same as $3 \times 5/8$, which we just studied.

Examples 3 and 4 are best illustrated in such ways as are indicated on page 255.

Example 5 follows from Example 4 by changing the order of the factors.

Example 6 may be illustrated as were Examples 3 and 4.

At this point the rule for finding the product of two fractions should be developed. Cancelling before multiplying should also be developed at this point.

Example 7 presents no serious difficulty and Example 8 follows by interchanging factors.

In actual practice Example 9 is solved in several ways and there seems to be no definite consensus of opinion as to the best method of doing it. The usual method is to reduce $3\frac{1}{2}$ and $2\frac{3}{4}$ to improper fractions, multiply, and reduce. That is, $3\frac{1}{3} \times 2\frac{3}{4} = {}^{10}/_{3} \times {}^{11}/_{4} = {}^{55}/_{6} = 9\frac{1}{6}$.

In examples where large numbers are involved the so-called four-step method is shorter. Thus, the product of $68\frac{3}{8} \times 54\frac{3}{4}$ is found as follows:

$$\begin{array}{r}
68\frac{3}{8} \\
\underline{54\frac{3}{4}} \\
\frac{9}{32} \\
51\phantom{\frac{9}{32}} \\
20\frac{1}{4} \\
272\phantom{\frac{9}{32}} \\
\underline{340\phantom{0\frac{9}{32}}} \\
3743\frac{17}{32}
\end{array}$$

First find the partial product

$\frac{3}{4} \times \frac{3}{8} = \frac{9}{32}$

$\frac{3}{4} \times 68 = 51$

$54 \times \frac{3}{8} = 20\frac{1}{4}$

Then multiply 68 by 54 as usual and add the products.

This method may be shown incidentally to some of the brighter children and they will like it.

146. Division of fractions. — Many methods have been used for explaining division of fractions. It is perhaps as difficult an operation to *explain* as any in the whole arithmetic. At any rate, the formula, "Invert the terms of the divisor and proceed as in multiplication," is clear in the memory of most of us, as a rule that pointed the way, but which gave not the least inkling of "why." Perhaps the best method of developing this principle so as to create a feeling of satisfaction in the minds of pupils possessed of more than usual intellectual curiosity (the others do not care, the rule satisfies them) is to develop systematically the idea: "To divide by a number is the same as to multiply by its reciprocal." Thus dividing by 2 is equivalent to multiplying by 1/2. This is well known to the child, for it is the way he has always multiplied by 1/2 (taken 1/2 of a number). Similarly dividing by 3 is equivalent to multiplying by 1/3 and so on. We can also develop the division of integers by such fractions as 1/2, 1/3, 1/4, etc. by showing that this amounts to the same as multiplying by 2, 3, 4, etc. These serve by analogy to make reasonable such statements as, "Dividing by 2/3 is equivalent to multiplying by the reciprocal of 2/3, or 3/2." (Two numbers are reciprocals, each of the other, if their product is unity.) Another method of developing division by fractions is illustrated by the following example:

Divide ¾ by $^2/_5$.

SOLUTION: Multiplying the dividend and divisor leaves the quotient unchanged. Hence we may multiply both by 4 × 5 (the product of the denominators).

Therefore $3/4 \div 2/5 = 15 \div 8 = 15/8$.

This method is clear to the child and may always be resorted to in case of doubt. As will appear in the next section it is the shortest and best method to use in actual practice when large compound numbers are involved.

After solving a few examples this way it may be pointed out that "the numerator of the result is the product of the numerator of the dividend and the denominator of the divisor, and the denominator of the result is the product of the denominator of the dividend and the numerator of the divisor." This is a long sentence containing many large words, but the substance of it is not hard to understand. Thus the teacher, pointing to 3 and 5 in $3/4 \div 2/5$, may say, "3×5 gives the numerator" and, pointing to the 4 and 2, "4 × 2 gives the denominator." After having repeated this process with several examples she may go back to the first one (all the examples having been left on the board) saying, "If we invert the divisor,[1] we get the same result simply by multiplying." After the examples on the board have all been gone over in this way, she may remark, "We will now try to get a short rule for dividing by a fraction. Suppose we try this one: "*To divide by a fraction invert it and then multiply.*" This rule is tried out on the examples just considered (still left on the board) and it is found that "it works." Other examples should then be given and the children asked to work them two ways, one according to the rule and one by the general method given above. In the course of ten or fifteen minutes the whole thing is settled, and, what is more, the rule has received considerable attention and will not readily be forgotten.

147. Division of mixed numbers. — The obvious method is to reduce dividend and divisor to improper fractions and then invert and multiply. However, the general method described in section 146 is shorter and simpler except when the numbers involved are very small.

To illustrate: Divide $18\frac{2}{3}$ by $4\frac{1}{2}$.
SOLUTION: $18\frac{2}{3} \div 4\frac{1}{2} = 112 \div 27 = 4^4/_{27}$.

Both dividend and divisor are multiplied by 6, which removes the fractions.

The following form may be used. 18

6×4 [illegible]

$$4\tfrac{1}{2}\,\overline{)\,18\tfrac{2}{3}} = 27\overline{)\,112} = 4^4/_{27}.$$

By the usual method we should have:

$$18\tfrac{2}{3} \div 4\tfrac{1}{2} = {}^{56}/_3 \div {}^{9}/_2 = {}^{56}/_3 \times {}^{2}/_9 = {}^{112}/_{27} = 4^4/_{27}.$$

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[1] "Turn it upside down" or "stand it on its head" may be used as alternative expressions at times in the verbal explanation. They serve to attract attention. A little flavor of fun in the schoolroom is excellent.

The shorter method leaves out the figures $56/3 \div 9/2 = 56/3 \times 2/9$, which is about half of the figures written in solving the problem by the longer method. That is, the shorter solution is about half as long as the other. It is believed worth while to have the children learn this shorter method inasmuch as little or no effort is required to understand it while at the same time their hold on a fundamental principle is strengthened.

148. Oral drill in fractions. — The fractions which actually occur in practical life are so simple that the operations on them may be carried out mentally (without making a written record to supplement the memory). The child should learn to add orally such fractions as $1/2 + 1/4$, $1/2 + 3/4$, $1/3 + 1/6$, etc. The first exercises should be such as are shown here in which one of the two fractions to be added may be reduced to the same denominator as the other. The following list gives all fourths and eighths that may be used:

¼ + ⅛, ¼ + ⅜, ¼ + ⅝, ¼ + ⅞, ¾ + ⅛, ¾ + ⅜, ¾ + ⅝, ¾ + ⅞.

Fourths and sixteenths, eighths and sixteenths, halves and eighths, halves and sixteenths, etc., may be used.

When the sums have been read the children may be asked to go over the same exercises and give the differences between the fractions in each pair instead of their sums. Later the same lists may be used to get products and quotients.

When these fractions have been mastered, pairs of fractions, like $1/2 + 1/3$, $1/2 + 2/3$, $1/3 + 1/4$, etc., may be added and later these fractions may be used for the other combinations.

Such oral work is of real importance. Not infrequently a graduate of a high school is unable to find the value of $1/2 + 1/3$, or $1/2 - 1/3$ without putting down all the work in formal detail, such as should be used only in case of fractions involving large and unwieldy numbers.

Such drill should be continued from time to time, possibly in several grades, until the combinations are thoroughly mastered. Whether or not the formal methods of adding and subtracting fractions (the division method for finding L. C. M., etc.) are learned does not matter much. Those who do not continue the study of mathematics beyond arithmetic will seldom if ever have need for these methods and those who continue into algebra will find a full treatment of the subject there. However, a section will be devoted to these methods here, if for no other reason than to illustrate one difficulty encountered by the makers of text-books.

149. Formal method for finding L. C. M. — The well-known process may be illustrated by finding the L. C. M. of 12, 16, 18, and 24.

$$\begin{array}{r|rrrr} 2 & 12, & 16, & 18, & 24 \\ \hline 2 & 6 & 8 & 9 & 12 \\ \hline 2 & 3 & 4 & 9 & 6 \\ \hline 3 & 3 & 2 & 9 & 3 \\ \hline & 1 & 2 & 3 & 1 \end{array}$$

The L. C. M. is $2 \times 2 \times 2 \times 3 \times 2 \times 3 = 144$. This process is *formally* simple, but it is not so easy to understand why it does what it is supposed to do.

It is easy, however, to devise a *direct* method for finding the L. C. M. of numbers which is readily rationalized.

Using the same numbers as above, and factoring each one, we have:

$$12 = 2 \times 2 \times 3,\quad 16 = 2 \times 2 \times 2 \times 2,\quad 18 = 2 \times 3 \times 3,\quad 24 = 2 \times 2 \times 2 \times 3.$$

To be a multiple of 12 a number must contain the factors 2, 2, 3; to be a multiple of 16 it must contain the factors, 2, 2, 2, 2; to be a multiple of 18 it must contain the factors 2, 3, 3; and to be a multiple of 24 it must contain the factors 2, 2, 2, 3.

Hence it is seen at a glance that the L. C. M. must contain the factor 2 four times and the factor 3 two times. That is, the L. C. M. is $2 \times 2 \times 2 \times 2 \times 3 \times 3 = 144$.

This may be seen at once without even writing down the factors, and the expression $2 \times 2 \times 2 \times 2 \times 3 \times 3$ may be written down directly by simply looking at the numbers 12, 16, 18, 24. This method is absolutely obvious and in practice much shorter than the usual one. Nevertheless, any possible explanation of it is lengthy and therefore "looks hard." That is enough to make it unavailable for most texts. The process given first in this section looks businesslike and can be described in very short formal rules. Let no one suppose that this is the only place in arithmetic where a simpler method is discarded simply because it does not readily lend itself to terse and crisp statements.[1]

150. The three problems in fractions. — Our textbooks on arithmetic contain many concrete problems involving fractions. The great majority of these, especially among the more difficult ones, will be found to require the solution of problems of which the following are type examples:

Type 1: 3/4 of 7/8 equals what?
Type 2: 21/32 is 3/4 of what number?
Type 3: 21/32 is what part of 7/8?

The verbal statement of a problem may serve to conceal, at least for a time, that it leads to one of these types.

These types may all be studied under the single form:

$$3/4 \times 7/8 = 21/32.$$

If the product is stricken out, we have Type 1 — a simple problem in multiplication. If the 7/8 is stricken out, we have Type 2. This is obviously a problem in division since the product, 21/32, and one factor, 3/4, are given (see section 189). Finally, if 3/4 is stricken out, we have Type 3, since the product, 21/32, and one factor, 7/8, are given. It is clear, therefore, that the three types given above are equivalent to finding the missing numbers in:

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[1] The people who sell books by counting lines used in each explanation and urging that book as the best whose explanations are the shortest have contributed much toward the undisputed superiority which America holds over the rest of the world in the elegance and simplicity of her school textbooks; but they have also contributed toward something else. The briefest statement is not always the greatest in merit. If it were, silence, utter and eternal, would be merit *par excellence*.

(1) $3/4 \times 7/8 = ?$, (2) $3/4 \times ? = 21/32$, (3) $? \times 7/8 = 21/32$.

The student should take comfort from the statement that if he masters these three problems fully he will seldom meet a problem in fractions which he can not solve, provided, of course, that he possesses sufficient power of attention to actually find out what the problem is. Section 192 should be noted in this connection.

REVIEW QUESTIONS

1. Discuss the introduction of fractions in the third grade. What kind of fractions should be introduced this early? How should the study of fractions be continued in the fourth grade? (§ 138.)

2. What fractions are found in common practical use? How may children obtain lists of such fractions? (§ 139.)

3. What is the fundamental property of fractions? Show how this property is used, (a) in reducing fractions to a common denominator; (b) in cancellation; (c) in dividing fractions. (§§ 140, 141, 146, 147.)

4. Discuss the use of objects in teaching fractions. What is the force of the objections made by some to the use of such objects as half an apple or one fourth of a pie to illustrate the meaning of fractions? (§ 142.)

5. Discuss the "carrying" method in subtracting mixed numbers. What is the advantage of using the same method of subtraction throughout the course? (§ 144.)

6. Give a list of types of examples in multiplication of fractions in the order in which they should be introduced. Consider the value of the "four-step" method of multiplying mixed numbers. (§ 145.)

7. Discuss the teaching of the division of fractions. How may the principle of section 140 be used to reduce a problem in division of fractions or mixed numbers to a problem in division of integers? How can we pass from this method to the usual rule for dividing by a fraction? (§§ 146, 147.)

8. How may simple fractions be reduced to a common denominator? Compare the value of this simple method with the usual *formal* method for reducing fractions to a common denominator. Discuss the value of continued oral drills on simple fractions. (§§ 148, 149.)

CHAPTER IX

Decimal Fractions

151. Early use of decimals. — In practically all courses of study decimal fractions are introduced informally before any direct study of their structure or meaning is possible. In writing dollars and cents the decimal point is used to separate them. Children are taught to add, subtract, multiply and divide such numbers, to carry across the decimal point, etc. In this early stage the multipliers and divisors are practically always integers.

Nothing is said at this stage about "decimals"; it is simply stated that the period, also called a "decimal point," is used to separate the dollars from the cents. The child learns to place the decimal point in the product and the quotient, the multipliers and divisors being integers.

152. Decimals and common fractions. — After defining decimals as fractions whose denominators are 10, 100, 1000, etc., the next step is to learn the decimal equivalents of some of the important simple fractions. The identities, 1/2 = 5/10 = .5, 1/4 = 25/100 = .25, should be noted with care. A small number of the important common fractions should be reduced to decimal equivalents and memorized. Decimals connect so directly with percentage and, in one form or another, are used so generally that well-equipped business men have them on the tip of the tongue. The reduction of common fractions to decimals and of decimals to common fractions will serve to make clear the meaning of both kinds of fractions and their identity of meaning though not of form. At this stage the children should be encouraged to bring to the class instances where decimal fractions are used. Representing dollars and cents will be noted as the most common and most important. It will also be found that the same numbers are frequently represented by both kinds of fractions. Thus the commercial sizes of wires are often given both ways. When this is discovered, the common fractions accumulated when these were first studied (see section 139) may be reduced to decimals. Indeed these lists of fractions will be the very best exercises possible for this purpose. It will be noted that for some of these fractions the process terminates while for others it apparently does not terminate. When the decimal becomes repeating, it may be pointed out that it can *never* terminate. In such cases the child may be directed to stop at three or four or five places as may be thought best. At any rate the children will be interested in reducing the diameters of wires to decimals, such as are given in any engineers' handbook. Such a book contains much information, simple enough to be understood by a layman, and a copy should be in the library of every school.

153. Reasons for using decimals. — Decimals are used for two reasons: (1) Numbers in the decimal form may be readily compared as to magnitude; (2) the fundamental operations on decimals are much more compact in form and less laborious than are the corresponding operations on common fractions.[1]

Children are interested in baseball scores, and it will be instructive for them to attempt to decide the standing of the teams in a league when they have given the number of games played and the number won and lost by each team. In many cases the children will find it practically impossible to rank the teams without reducing to a decimal the ratio of the number of games won to the number of games played by each team.

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[1] "The importance of decimal fractions is recognized more to-day than ever before and the subject is one of the most important in arithmetic." Brown and Coffman, *op. cit.* (p. 31), p. 212.

It will not be difficult to find appropriate illustrations to show that adding fractions, for instance, may be very tedious if not practically impossible, while the decimal approximations may, of course, be added readily.

154. Approximate decimals. — Approximation is strange to the child. His number concepts are derived largely from counting, and counting is exact (see section 167 in Chapter X). It therefore seems absurd to him to use numbers which are not "exactly right" but only "nearly right." In practical affairs, decimal approximations are used for several reasons: (1) Not all common fractions can be expressed exactly as decimals, and still it is desirable to use decimals. (2) In many cases only approximations are needed. Thus, for certain purposes it may be sufficient to represent 5/32 by .156 though 5/32 may be represented exactly by .15625. (3) When numbers are used to represent measured magnitudes (for definition of measurement see section 166) they must in the nature of the case be approximations. We are here concerned only with the difference between a common fraction and its approximating decimal and with the difference between two decimals which are identical to a certain number of decimal places.

The last figure in an approximating decimal is such as to make the approximation more nearly exact than the next higher or the next lower figure would make it. Thus, .16 or .156 or .1563 may be used as approximations of .15625. The decimal .16 means that the number approximated is less than .165 and equal to or greater than .155. Note that instead of .155 we write .16 instead of .15, which is equally near the correct number. Similarly .1563 and not .1562 is used to approximate .15625, though the latter is equally correct. The decimal .156 means that the number approximated is less than .1565 and greater than or equal to .1555. Again, .1563 means that the number approximated is less than .15635 and greater than or equal to .15625.

If a number lies between .155 and .165, it can not differ from .16 by more than .005. Hence an approximation to two decimals can not differ from the number approximated by more than .005 or 1/200. If a number lies between .1555 and .1565, it can not differ from .156 by more than .0005. Hence an approximation to three decimals can not differ from the number approximated by more than .005 or 1/2000. Similarly an approximation to four decimals can not differ from the number approximated by more than 1/20000.

If two decimals are identical in the first two places, as for instance .16, then both are less than .17 and neither can be less than .16. Hence two decimals identical to two places differ by less than .01. Similarly two decimals identical to three places differ by less than .001. These matters should be made clear as they are fundamental in considering the much neglected question as to the number of decimal places to which computations should be carried. It is not infrequent in American schools to see computations carried to five, six, and even eight or ten decimal places though it may well be that not more than two places have any meaning (see section 164).

155. Approximate integral numbers. — Many integral numbers are only approximations. The banker knows *exactly* how many dollars and cents he has in his vault at a certain time, but no one knows *exactly* how many people there are at any one time even in a moderate-sized city. The population statistics of the census are only approximations. While the census is being taken, there are numerous births and deaths. If the city is large, the population may not be known within several hundreds or even within several thousands. It is doubtful if the United States census contains information accurate within a hundred thousand. No one knows within wide limits the total wealth of a country or its total income. Indeed an individual who has at all varied and extensive property interests does not know except within certain limits the total value of his property. It may even vary largely from day to day or with different theories as to what should be regarded as property. Numbers representing such uncertain quantities should show the degree of uncertainty by zeros in the last places to the right. Thus the statement that a man is worth $140,000 should be taken to mean that he is worth less than $145,000 and not less than $135,000. The census reports give populations of cities, states, and countries *exactly*, but such figures are misleading. The census of 1920 gave the population of the United States as 105,708,771 though 105,700,000 no doubt represents the facts more accurately. The five zeros to the right indicate that the actual number lies between 105,750.000 and 105,850,000. Hence, 105,700,000 is supposed to differ from the actual number by not more than 50,000. Possibly 106,000,000 would be a better number. The six zeros would indicate that the number is supposed to be known within 500,000. That is, the number is accurate to 1 in 200. Such ideas as these are essential in reading numbers understandingly, and the child has a right to be instructed in them at some stage of his common-school education. The time when decimal approximations of numbers are considered would seem to be the most opportune for doing this work.

156. Addition and subtraction of decimals. — There is no necessity for attempting to rationalize again the process of carrying in the addition and subtraction of decimals unless indeed it be mainly for the purpose of reviewing the nature of these processes in general. Hence all that is necessary in teaching addition and subtraction of decimals is to make sure that the decimal points are placed in a column and possibly to add zeros so as to make all decimals of the same order. The chief danger at this point is overexplanation.

157. Multiplying a decimal by a one-figure integer. — The first point to be attended to in the multiplication and division of decimals (as in the study of most topics in arithmetic) is the analyzing of the subject so as to isolate each element of difficulty for separate study and drill. The multiplication of a decimal by a one-figure number is obvious, but it should be studied separately before any other forms are brought in.

To illustrate: In the example in the margin 6 × 8 (thousandths) is 48 (thousandths). Write the *8 directly under the 8* in the multiplicand. Proceed as in ordinary multiplication and place the decimal point in the product *directly under* the decimal in the multiplicand.

$$\begin{array}{r} 42.078 \\ 6 \\ \hline 252.468 \end{array}$$

158. Multiplying and dividing a decimal by 10, 100, 1000, etc. — That a decimal may be multiplied by 10 by moving the decimal point one place toward the right follows in precisely the same manner as the multiplying of an ordinary number by 10 by annexing a zero to the right. This whole matter may be reviewed at this time, showing that either process puts every digit into the next higher order and that hence each digit is made to represent a number 10 times as great as before. Multiplying by 100 and by 1000 should be explained in the same manner. The opportunity presented here to acquire a better understanding of the whole process and of the nature of the decimal notation should not be missed.

Division by 10, 100, etc., is explained in a precisely similar manner both as regards moving the decimal point and striking off zeros. Show that moving the decimal point makes possible division by 10, etc., of such numbers as 486 which can not be divided by striking off zeros. The child will be interested and pleased to find that he is learning methods that will work where those he knew before failed.

159. Multiplying a decimal by a one-figure decimal. — We have learned that to multiply by 30, for instance, we multiply by 3 and then by 10. Similarly to multiply by .3 we multiply by 3 and then by .1 or, what is the same thing, divide by 10. Thus, in the example in the margin we multiply by 3 in the usual manner and then put the decimal point one place farther to the left than we would if the multiplier were 3. Multiplication by .03 or .003 may now be explained in the same manner. The similarity to the process of multiplying by 300 or 3000 should be noted. This will help to gain a firmer understanding of multiplication in general and will help to explain and render satisfactory to the child the rule for placing the decimal point in multiplication. We now easily find the rule for placing the decimal point in the product; namely, *the number of decimal places is equal to the sum of the numbers of decimal places in the multiplier and multiplicand.*

$$\begin{array}{r} 806.42 \\ .3 \\ \hline 241.926 \end{array}$$

160. Multiplying a decimal by a decimal. — We are now in a position to multiply two decimals and to place the decimal point in the product without stating any further rule. In the example in the margin we first multiply by .03. By the rule developed in section 159, the decimal point goes between the 4 and the 1. By the same rule we know that multiplying by .4 gives a three place decimal. Hence, in order to get the decimal points in a column, we must start writing this product directly under the 4 of the multiplier.[1] On completion of the work the

$$\begin{array}{r} 715.84 \\ 2.43 \\ \hline 21.4752 \\ 286.336 \\ 1431.68 \\ \hline 1739.4912 \end{array}$$

---

[1] "Multiplication of decimals may be explained also in the following manner. Required to multiply .043 by .05. If we multiply .043 by 5, the result is .215, but this example differs from the given example only in that the multiplier is 100 times as large as the given multiplier

usual rule is clearly apparent. All that is necessary is to state it. When the rule has been developed the decimal points in the partial products are, of course, omitted.

161. Division of a decimal by an integral number. — The work of division is, of course, carried out exactly as with integral dividends. Placing the first figure of the quotient directly above the last figure of the first partial dividend and placing the decimal point of the quotient directly above the decimal point in the dividend takes care completely of the problem of locating the decimal point in the quotient.

By means of examples the analogy with the division of integral numbers should be pointed out. The first partial dividend is 571 *tens* and hence the first figure of the quotient represents tens and is placed in tens' place. This serves as an opportune review of ordinary long division and renders the placing of the decimal point simple and clear. No *rule* for placing the decimal point is necessary.

$$\begin{array}{r} 12.67 \\ 451\,\overline{)\,5716.24} \\ \underline{451} \\ 1206 \\ \underline{902} \\ 3042 \\ \underline{2706} \\ 3324 \\ \underline{3157} \\ 164 \end{array}$$

162. Division of decimals in general. — By the simple device of multiplying the dividend and divisor by the same number any problem in division of decimals may be reduced to the case where the divisor is an integral number. (Compare this with sections 140, 147.)

Thus to divide 1561.842 by 1.96, we proceed as shown in the solution given here. The first step is to move the decimal

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(since 5 equals 100 × .05). Therefore the product is 100 times too large, and the correct product is 1/100 of .215, or .00215.

"After the rule for multiplication has been rationally developed, the pupil should use it instead of developing the principle each time he wishes to find the product of the two decimal fractions. The first essential is to make the process seem reasonable to the pupil, and the second is to acquire facility in the application of the process." Brown and Coffman, *op. cit.* (p. 31), p. 204.

point two places to the right in both dividend and divisor; that is, to multiply each by 100, thereby making the divisor an integral number. The work of dividing and placing the decimal point is then exactly as in section 161. As in that case too, no rule for placing the decimal point is necessary. This is particularly fortunate since the usual rule is likely to cause errors. This is especially the case when zeros are annexed to the partial dividends as the division proceeds. If, however, it is desired to teach the rule, it may readily be obtained from the work just described.

```
             796.85
   1|96. )1561|84.2
          1372
           1898
           1764
            1344
            1176
             1682
             1568
              1140
               980
               160
```

In the above example it should be pointed out that the remainder (not regarding its decimal character) is greater than one half the divisor and that hence the quotient should be written 796.86.

163. Number of decimal places in numbers to be added. — If it has been decided that a sum of several numbers should be correct to a certain number of decimal places, it may be necessary to have the numbers to be added correct to one or possibly two more decimal places. As an illustration of this it may be best to reduce several fractions to decimals and add them and then add the fractions as common fractions and reduce the sum to a decimal. Thus, if the fractions 1/2, 1/4, 3/8, 3/16 are reduced to two-place approximations according to the rule and added, we get 1.32, whereas adding the fractions first and reducing gives 1.3125, and the decimals are not identical to two places. If the fractions are reduced to three place decimals and added, we obtain 1.313, which is identical with 1.3125 to two places.

164. Degree of accuracy of the product of decimals. — We have already seen that a decimal like 1.4, in case it is an approximation, indicates that the true value of the number is less than 1.45 and not less than 1.35. Similarly the number 2.8 indicates that the true value of the number is less than 2.85 and not less than 2.75. Hence we know that the true value of the product is less than 4.1325 and not less than 3.7125, and that is *all* we know about it (see examples to the left). If, however, we multiply 1.4 by 2.8, we get 3.92. If this were known to be an accurate two-place approximation, we should *know* that the product lies between 3.915 and 3.925, which as a matter of fact we do not know.

$$\begin{array}{r}1.35\\ \underline{2.75}\\ 675\\ 945\\ \underline{270}\\ 3.7125\end{array}\qquad\qquad\begin{array}{r}1.45\\ \underline{2.85}\\ 725\\ 1160\\ \underline{290}\\ 4.1325\end{array}\qquad\qquad\begin{array}{r}1.4\\ \underline{2.8}\\ 112\\ \underline{28}\\ 3.92\end{array}$$

This example shows how futile it is to retain a large number of decimal places when the multiplier or multiplicand or both are approximations. Without further investigation we will say here that a greater number of decimal places in the product than the larger number of such places in multiplier or multiplicand is not justified. Great efforts have been made by teachers in our universities and technical schools to get those who control the lower schools to take some account of this matter — at least in secondary schools. So far the effort seems to have had little effect. Recently an instructive episode took place in a class in arithmetic for forest rangers. The class was composed of some twenty-five young men, all of them graduates of our common schools and some with considerable high-school education. Among them was a foreigner, educated in the common schools of Switzerland. He was not more intelligent than the others, perhaps not even above the average. These American-educated youths worked faithfully, carrying decimals in products to ten places or more, while the young man who was educated in Switzerland cut off the decimals where they should be cut off. The campaign for improvement in this respect has been so persistent and so futile that it is not deemed worth while to go into the matter further here. It will be regarded as "too hard" to be made part of our curriculum of "easy tasks" — a curriculum made tiresome by too much of what is essentially clerical work of the routine type. The teacher who is sufficiently interested will have little trouble in finding references on this topic; and she will easily be able to work it out for herself. It is not difficult. The brighter pupils will find it interesting, and the time saved by omitting unnecessary and meaningless figures will more than compensate them for their trouble.

165. Omitting figures in dividend and divisor. — Closely allied to the topic of the preceding section is the matter of omitting parts of the dividend and divisor in case these are approximate numbers. The total wealth of the United States has lately been estimated at 260 billions, and our total population at 105 millions. To find the average wealth, we need, therefore, to carry out the division indicated here. It is at once clear that six zeros may be stricken out from both dividend and divisor, giving the second division problem. When we consider that the total wealth is surely not known within billions and the population not within a half million or so, we may come to the conclusion that 260,000 ÷ 100 = 2600 is as near the average wealth in dollars as we can get it. At all events the person who attempts to get the average wealth by using as divisor the number of people expressed in nine significant figures shows a sad lack of insight into matters of this kind. This is one of the most discouraging topics to consider, and it may be best not to pursue it further (see section 155).

$$105,\cancel{000},\cancel{000}\,\big)\,\overline{260,000,\cancel{000},\cancel{000}}$$

$$\begin{array}{r@{\,}l}
 & \underline{2476} \\
105\,\big) & 260,000 \\
 & \underline{210} \\
 & 500 \\
 & \underline{420} \\
 & 800 \\
 & \underline{735} \\
 & 650
\end{array}$$

REVIEW QUESTIONS

1. How are decimals used in the third and fourth grades? (§ 151.)

2. Give several reasons why decimals are used instead of common fractions. What drill would you give in reducing fractions to decimals? (§§ 152, 153.)

3. Discuss approximate decimals. If a quantity is represented by 4.7, between what values is it asserted to lie? If decimals are used to represent measured quantities will they be approximate or exact? (§§ 154, 167.)

4. Give examples of integral numbers which are known only approximately. How should such numbers be written? (§ 155.)

5. How should addition and subtraction of decimals be taught? (§ 156.)

6. Give the steps in multiplication of decimals in the order in which they should be introduced. How is the rule obtained for placing the decimal point in the product? (§§ 157–160.)

7. Give the steps in division of decimals in the order in which they should be introduced. How may the formal rule for placing the decimal point in the quotient be avoided? Show how the use of this rule is likely to introduce errors. How may the fundamental principle of fractions be used in division of decimals? (§§ 161, 162, 165.)

8. Discuss the accuracy of the results obtained by adding, subtracting, multiplying, or dividing approximate decimals. (§§ 163, 164.)

CHAPTER X

MEASUREMENT AND DENOMINATE NUMBERS

166. Meaning of "measurement." — Measurement is the process of determining the magnitude of a quantity by finding how many times a fixed unit of measure is contained in it. The process consists in dividing the quantity to be measured into parts, each equal to the unit of measure, and then counting the number of these parts.[1] According to this definition ordinary counting does not constitute measuring though counting is a part of every act of measuring. Counting the number of people in a crowd would not be regarded as "measuring" the crowd since there is no act of dividing it into parts, each equal to a unit of measure. The parts counted exist as separate objects independent of any act of him who counts, and moreover these objects are unequal in nearly every respect.

The question as to the meaning of the word "measurement" is of some importance. No little confusion has arisen because certain writers, prominent among whom are McLellan and Dewey,[2] have used the word so as to include ordinary counting. It is of course true that, considering logical requirements only, a writer may adopt any definition he chooses provided he states it explicitly and then adheres to it throughout. However, in case an unusual definition is adopted, there is no small danger that the writer may himself at times lapse into the current usage,[1] thus causing confusion both for himself and the reader; and it is almost inevitable that the reader will forget the special definition and interpret the words in their usual meaning. Indeed, many who quote an author's statements have not read the context. A writer who uses words in an unusual sense is certain to be misunderstood perhaps by a majority of those who seek to follow him. This would appear to be the case with many followers of McLellan and Dewey, and that including some who would naturally be supposed to read and interpret with care. McLellan and Dewey maintain that the child gets his idea of numbers from "measuring" as the word is used by them. Some of their disciples try to have children get their number concepts from measuring as commonly understood. Through the minds' eyes of these people, we see the little fellow of six, advancing upon the world with his foot rule, measuring all things great and small to learn the meaning of one and two and of two and three. One series of arithmetics written for this abnormal child contains profuse acknowledgments to McLellan and Dewey's *Psychology of Numbers*. This shows how dangerous it is to write in a partly foreign tongue.

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[1] The statement about a "fixed unit" and division into "equal parts" must be taken with a grain of salt since no measurements are "exact." See section 167.

[2] See citation from these authors given on page 31.

167. Measurements always approximate. — Nothwithstanding frequent remarks about exact measurements it is true that measurements are *never* exact.[2] This fact must be made clear at some stage in the course and may be made to appear even in the lower grades. Suppose a line about sixteen inches long has been drawn on the board. The answers to the following questions may then be obtained by using the yardstick. Is the line more or less than 12 inches long? Is it more or less than 20 inches?

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[1] The writer once examined twenty different texts on plane and solid geometry with reference to the meaning of the words "circle" and "sphere." In only one of these books were both of these words used consistently throughout.

[2] The chief practical distinction between measuring and counting is that the latter is *always* exact while the former *never* is. There are exactly twelve persons on a jury, but we know only approximately the length of time they deliberate to reach a verdict.

Is it more or less than 14 inches? Is it more or less than 18 inches? Is it more or less than 15 inches? than 15½ inches? Is it more or less than 17 inches? than 16½ inches? So far the questions will be answered with certainty and unanimity. If it is asked whether the line is more or less than $15^{31}/_{32}$ inches or $16^{1}/_{32}$ inches, the answers are likely to be neither certain nor unanimous. That is, the line is known to be *approximately* 16 inches long, and it is known with certainty that its length is between 15½ inches and 16½ inches. Of course the child will find the length of the line without the aid of such questions. He applies the yardstick to the line and then announces that it is 16 inches long, omitting the word "about." But sooner or later this "about" becomes absolutely essential. Questions such as those given above will serve to bring out the nature of all measurements; namely, that the information furnished by them is that the measured magnitude is greater than a certain number of units and less than a certain other number. There is always an interval of uncertainty. Increasing the accuracy of the measurement simply serves to make this interval smaller.

In announcing the result of a measurement a single number is usually used, but there is always a tacit understanding (on the part of those who know what they are doing) that this result is an approximation within certain limits. For practical purposes it is often necessary to know the *degree* of accuracy. There are few simple elements of great practical value which are more neglected in our courses in arithmetic than is this.[1]

168. Denominate numbers. — A denominate number is always given in terms of one or more definite units of measure which have been established either by law or custom. It should be clearly understood that not all concrete numbers are denominate numbers. In the expressions, "15 hogs" and "4 snowstorms," we are using concrete numbers but not denominate numbers. On the other hand all denominate numbers are concrete numbers.

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[1] In practice, in giving results of measurements the degree of possible error is often given. In ordering articles from a dealer or a factory dimensions are often given and the closeness with which the measurements must be followed depends largely upon the character of the object and its proposed uses. An order for a hundred feet of rope may be satisfactorily filled if the rope does not differ from a hundred feet by more than 6 inches, while a piston for an automobile must be accurate within, say, .0001 of an inch.

If more than one unit is used in one denominate number, these units usually represent magnitudes of the same kind. Thus distance may be given in miles and feet; and time, in days, hours, and minutes. There are some numbers which may perhaps be called quasi-denominate numbers which are given in two or more units of different kinds. Thus speed may be given in *miles per hour* or in *feet per second*. The amount of freight carrying done by a railroad or a truck may be given in *ton-miles;* and the power of an engine in *foot-pound-seconds*. The passenger traffic on a railway is given in *passenger-miles*, in which case one of the units (passenger) is not of the kind specified in the definition given above.[1]

169. What tables of denominate numbers should be learned. — The relations among the units used for various measurements are recorded in the tables of denominate numbers. It is frequently said that no person knows *all* the tables of denominate numbers used in the United States and this is possibly true. A glance at a large dictionary under the head "measurement" will be illuminating. Only a fraction of such tables is ever given in any set of elementary arithmetics. It is now agreed that not even all the tables given in a standard textbook should be committed to memory with the same degree of thoroughness. The relations among certain units should be learned as thoroughly as are the fundamental combinations. Among these are:

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[1] It may be interesting to give an account of the historical development of some of our more important units of measure. Any good encyclopedia contains full information. The following is from Brown and Coffman, *op. cit.* (p. 31), p. 174.

"About the middle of the nineteenth century the British Government adopted the length of a second's pendulum oscillation as the invariable unit of length. Of the 39.1393 parts into which it was divided the British yard contains 36. Each of these is the British inch. The distilled water which fills a cubic inch is found to weigh always the same amount under the same conditions. This weight is divided into 252.422 equal parts; the British pound avoirdupois contains 7000 of these parts.

"Again, it is possible to give the quantity of distilled water which weighs exactly 10 of these pounds avoirdupois. This quantity of water always fills the same space. The space is the 'content' of the British *gallon*, the unit of capacity. With the final precaution that the gallon measure shall be circular at base, not rectangular, and of given diameter, our units of capacity and weight are clearly connected and rest upon the firm basis of the oscillation unit of length."

For a more complete statement see the same book, pp. 173–179.

Length: The inch and foot, inch and yard, foot and mile. The farmer should also know the relations of the rod to the foot, yard, and mile.

Area: The relations between the square inch and the square foot; the square foot and the square yard; the square rod and the acre; the acre and the square mile.

Volume: The cubic inch and the cubic foot; the cubic foot and the cubic yard.

Capacity: The pint, quart, and gallon; the gallon and the barrel (all liquid measure); the pint, quart, peck, and bushel (all dry measure).

Weight: The ounce and pound; the pound and ton (avoirdupois).

United States Money: All standard coins.

Time: The second, minute, hour, day, week, month, year, leap year, — these not to include such relations as second and hour, minute and day, or hour and week.

Temperature: The Fahrenheit thermometer.

The child should also learn that there are certain other tables and the purposes for which they are used. Thus he should learn about the existence and uses of apothecaries' measure, apothecaries' weight, troy weight, and the carat. He should also learn where to look for such tables in case of need. Problems involving such other tables may well be assigned, the pupil being free to consult the tables as he needs them.

170. Special units — approximate value of units. — The child who lives on a farm should know about the *hand*, a unit 4 inches long used to measure the height of horses. The fact that shoes which differ by one *size* differ just ⅓ of an inch in length is of interest to all. There should be discussion of the *nautical mile* which is roughly 1.15 ordinary miles (statute miles), of the *square* used in measuring roofing, and of the *board foot* used in measuring lumber.

There are many approximate relations in common use which will be of interest and value, some of them particularly to those living on the farm. Such are:

One bushel = 1.25 cubic feet.
One cubic foot = .8 bushels.
One heaped bushel = 1.6 cubic feet.
One ton stove coal (hard) = 35 cubic feet.

For children who live on the farm tables should be available showing the contents of silos in tons. For them it will also be interesting to know that approximately:

One ton of timothy hay in well-settled mow = 450 cubic feet.
One ton of clover hay in well-settled mow = 550 cubic feet

The farmer (and certainly the surveyor) may need to know that there are 43,560 square feet in one acre.

171. The uses of various units. — Along with the study of the various units the child should learn the uses to which they are put. Thus the dimensions of a sheet of paper are measured in inches; the dimensions of a room, in feet; the dimensions of a field, in rods; the length of a running track, in yards or in parts of a mile; and the distance between cities, in miles.

The children may be asked to make lists of articles measured by the pint, the quart (dry and liquid), the peck, the bushel, and the barrel. Similar lists should be made for all the units as they are studied. This will serve not only to develop fuller ideas of the magnitudes of the units but also of their practical significance.

172. Reduction of denominate numbers. — Not infrequently it is necessary in practice to reduce a denominate number to a different unit. The length of a room is measured in feet but the carpet for it is bought by the yard. The width of a board is given in inches; but for the purpose of finding the number of board feet in it, it may be best to reduce the width to feet. The length of a field is given in rods while it is frequently measured by means of a steel tape marked in feet. The contents of a bin is first obtained in cubic feet while its capacity in bushels is what we want. The dimensions of a field are frequently measured in feet and the area found in square feet, but the farmer wants to know how many acres it contains.

Obviously no rules are needed for such reductions. All that is needed is the relation between the two units involved and such a modicum of native common sense as is necessary for all who are to profit by a course in arithmetic. Nothing is more deadening or more distasteful than superfluous rules.

173. The fundamental operations on denominate numbers. — Addition and subtraction of such numbers are the operations that are most likely to occur in practice. Such additions as are likely to occur are very simple and will present little difficulty except to the very backward. Subtraction is more difficult and is of additional interest because the question has been raised as to whether the method of carrying (see section 126) can be made to work simply and naturally in this case. The answer can be given best by solving an example:

A man, born October 21, 1843, dies February 7, 1917. Find his age in years, months, and days at the time of his death.

SOLUTION: 1917 yr. 1 mo. 7 da.
1843 yr. 9 mo. 21 da.
73 yr. 3 mo. 16 da.

Add 30 days (1 mo.) to the 7 days. Then 37 da. − 21 da. = 16 da. Carry 1 month. Add 12 months (1 yr.) to the 1 month. Then 13 mo. − 10 mo. = 3 mo. Carry 1 year. Finally 1917 yr. − 1844 yr. = 73 yr.

Hence it is seen that in this case also this method is at least as simple and natural as the other methods.

Examples in multiplication and division of denominate numbers, such as still occur in our texts, though in diminishing numbers and in increasing simplicity, are seldom met in practical experience. If such problems are solved at all, they should be worked entirely without rules and with very little guidance. Some pupils in the class will get them, thereby experiencing the satisfaction that comes from conquering a difficulty. Others will fail, but for them the solution under detailed guidance would mean little more than the carrying out once more of the fundamental operations on ordinary numbers. They would learn little or nothing of value.

174. The metric system. Foreign money. — While it is quite generally agreed that an exhaustive study of the metric system of weights and measures should not be made in our elementary schools, there are many reasons why it should not be omitted entirely. Smith[1] enumerates these reasons as follows:

"(1) General information requires us to know a system used by a large part [It would be correct to say 'practically all'] of the civilized world, excluding the English-speaking portion; (2) It is used in all scientific laboratories in America; (3) American people should be sympathetic with a system that is likely to replace our own before long in all matters relating to our growing foreign trade; if we sell machines abroad, the measurements must be metric in most cases, and to foster this trade many of our skilled workmen will eventually need to use these instead of the awkward ones with which we are familiar."

If by an exhibition of courage and common sense we were to do what has been done in many countries; namely, make the metric system the only legal system, much temporary inconvenience would be caused (for one thing, all texts on arithmetic would have to be rewritten), but untold benefits would accrue and that at no distant date. The course in elementary arithmetic could be shortened by at least one year and the saving in the work of the world of business and the practical arts would be tremendous.[1]

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[1] *Op. cit.* (p. 111).

175. Foreign money. — Tables of foreign money should not be memorized. On the other hand a child attending American schools should be given the opportunity to learn that the *pound sterling* is used in Great Britain, the *franc* in France, the *mark* in Germany, etc., and to learn also the approximate values of these when at par. The pound is worth a little less than $5; 5 francs, a little less than $1; and 4 marks (when at par), a little less than $1.

The argument for the inclusion of this topic is not based upon the possible need of such information which the child may some day have when traveling in foreign countries but upon the general value of certain information which every fairly intelligent person has and the lack of which has come to be regarded as crass ignorance. The "practical" value of knowing that Peary reached the North Pole may be difficult to point out, but what sensible parent would want his child placed in a school where this would never be mentioned?

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[1] "The persistence of custom will delay its ultimate triumph for many years, but the best must eventually triumph. Since its adoption in France in 1840 it [the metric system] has spread with great rapidity. It is to-day either obligatory or permissive in every civilized country of the world. It is in general use among all civilized nations except England and the United States. Instruction in it is obligatory in all English schools. It has been the legal system in the United States since 1866; its use is required in some of the departments of our government, and is authorized in others. . . .

"It is desirable that teachers should appreciate the merits of the metric system. No teacher can afford to be ignorant of a system that has been so generally adopted by civilization and is the basis for all scientific work. A well-educated person to day should know the metric units in common use." Brown and Coffman, *op. cit.* (p. 31), pp. 284–285.

For an historical account of the metric system, see any good encyclopedia or this same book by Brown and Coffman, pp. 282–289.

What is said in section 169 about solving problems involving tables not to be memorized applies to possible examples involving foreign money.

176. The pupil's text as a permanent reference book. —This point would seem as appropriate as any other in this book for making certain remarks which the writer believes are of vital importance. Few statements are made with greater assurance by those who discuss educational questions than this: "One of the important aims of education is to learn where facts may be found when needed." Presumably the texts studied in our schools deal with facts that are more likely to be of subsequent use or interest than other facts which are omitted. No one pretends that even a reasonable fraction of the facts learned even in our elementary grades will be remembered permanently. There is probably no person living (not even the authors of textbooks while they are writing them) who remembers all that is contained in a good series of school geographies. For these reasons the child should be encouraged to keep his textbooks as permanent reference books. We all know and deplore the student in the college who figuratively says to a course as soon as he has finished it, "Get thee behind me," and rushes to the secondhand store to sell his book for the price of an admission to the movies. The system of furnishing free texts, which is very wide spread and still seems to be spreading, not only fails to encourage the pupils to keep their books as reference books but makes it practically impossible for them to do so. The book belongs to the school and must be surrendered when the subject is finished. If the school district were to make the pupils presents outright of the books, well and good, but loaning them to the pupils for the time of actual use in the school and then taking them away would seem to encourage one of the most deplorable tendencies in our education.

This criticism is not answered by saying that those who wish to buy and keep their books are at liberty to do so; nor is it answered by saying that in any event only a few would keep their texts for reference. It is the business of the school to assume educational leadership over the child, to point out the path, and to provide such conditions as will make it likely that the greatest possible number will follow it. Such difficulties as the greater cost when books are bought singly by each individual or the inability of some to buy their own books will be overcome readily enough when it is realized that a vital educational principle is at stake. For a further discussion of this topic, see section 324.

REVIEW QUESTIONS

1. Describe what is meant by "measurement." How does measurement differ from counting? (§§ 166, 102.)

2. Show that measurement is always approximate. To what extent is it possible and advisable to teach the approximate character of measurement in the early grades? (§ 167.)

3. What is meant by "denominate numbers"? Name some denominate numbers which are given in more than one unit. (§ 168.)

4. What tables of denominate numbers should be committed to memory? In how far should the uses of the various units be stressed? What approximate units should be taught? To what extent do the answers to these questions depend upon the character of the neighborhood in which the school is located? (§§ 169, 170, 171.)

5. Discuss the importance of the reduction of denominate numbers and also of the fundamental operations on them. To what extent should these be carried? (§§ 172, 173.)

6. What are the arguments *pro* and *con* for teaching the metric system and tables of foreign money? To what extent should these subjects be carried if introduced at all? (§§ 174, 175.)

7. Discuss the value of the textbook as the pupil's permanent reference book. What is the bearing of this point on the question of free textbooks? (§ 176.)

CHAPTER XI

PERCENTAGE

177. Percentage related to preceding topics in arithmetic. — Percentage introduces *nothing* that is new in the way of mathematical principles. This is fully recognized by all who have written on the teaching of arithmetic [1] and is surely clear to all who have produced school textbooks. It is not so certain, however, that it is equally clear to all who teach the subject or to all who supervise the teaching of it. Yielding no doubt to what is believed to be the demand of some teachers and school superintendents, not a few writers of texts have treated percentage as if it were based on principles peculiar to it and quite distinct from anything that precedes.[2] That this is a mistake both from the pedagogical and the practical standpoint there is not the slightest doubt. Recent texts show less of this tendency.

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[1] "The subject of percentage is almost wholly a language lesson. No new mathematical principles are involved in any of its applications." Brown and Coffman, *op. cit.* (p. 31), p. 213.

"In his treatment of percentage the teacher should constantly bear in mind that no new process which requires special development is under consideration." Walsh, *op. cit.* (p. 176), p. 228.

"(In beginning percentage) the children should not be aware that a new division of their work has been introduced." Paul Klapper, *The Teaching of Arithmetic*; D. Appleton and Company, 1916, p. 254.

"The child should see that 'per cent' is only another name for hundredths and that we may read 0.06 either 6 hundredths or 6 per cent." Smith, *op. cit.* (p. 111), p. 177.

[2] "In some textbooks on arithmetic percentage is treated as if it were a special process involving certain distinctive principles. . . . This elaborate treatment seems to be a mistake on both the theoretical and practical side." McLellan and Dewey, *op. cit.* (p. 31), p. 279.

"In teaching percentage most teachers present the topic as though it were a new branch of arithmetic, only remotely related to what the children already know." Klapper, *ibid.* p. 254.

The only new element in percentage is that hundredths, called "per cents," are used systematically as a basis of work. The fact that this is the only new element should be made clear at the outset and the pupils should never be allowed to lose sight of it. The pupils know for instance how to find six hundredths of a number and therefore they know how to find six per cent of it. Further rules or directions at this point are worse than useless. If a child has difficulty in finding six per cent of 2400, for example, ask him to find six hundredths of it; and when he has done so, lead him to see that he has just found six per cent of the number. This will amount simply to a repetition of the statement that six hundredths and six per cent are identical — that "per cent" means hundredths.

The following three examples represent all the essentially different problems in percentage:

1. Find 6 per cent (hundredths) of 800.
2. 48 is how many per cent (hundredths) of 800?
3. 48 is 6 per cent (hundredths) of what number?

It will be seen at once that these examples are identical in substance with the three problems in fractions on page 263.

If the children have trouble with percentage let them go back to the corresponding problems in decimals or in common fractions; and if they have learned these properly, the trouble will vanish. If they can not handle fractions and decimals, review as much of these as may be necessary but do not give "rules."

If the solution of problems has been developed in some such manner as is suggested in Chapter XII (pp. 294–328), it will be practically impossible for a child of fair intelligence to have real trouble with problems in percentage unless these are stated purposely in an involved manner to create trouble. That, however, is now out of fashion.[1]

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[1] We have come to realize that even at the best there is enough to be mastered to make the strongest exert himself, provided we are willing to depart a little from the kindergarten fringe of our subjects. It is no longer necessary to pull on our boot straps to get exercise.

178. Elementary drill in percentage. — While it is true that all of percentage is included in fractions, common or decimal, and that in case of essential difficulty we should fly to these, it is also true that to get a direct insight into percentage as an instrument of thought a good deal of simple drill is necessary. We should have oral drills like these:

Find 5% of 200, of 400, of 800, of 1000, of 1600.
Find 8% of 100, of 300, of 500, of 600, of 800.
50 is how many per cent of 100? of 200? of 400? of 1000?
10 is 5% of what number? 10% of what number? 100% of what number?
60 is 20% of what number? 10% of what number? 5% of what number?

Such drills should continue from the time percentage is first introduced until the pupils can think as readily in terms of per cents as in terms of ordinary units. Drills in the reduction of fractions to per cents should also be given from time to time.

179. Uses of percentage. — The subject of percentage and its applications extends over about two years of work in arithmetic. Hence there should be no trouble in finding illustrations of its uses. As each of the main applications is taken up, the child's idea of the uses of percentage will be enlarged and this might seem sufficient. It will be interesting, however, to have examples of the practical uses of percentage at the very beginning of its study. As in other similar cases do not give the pupils a large ready-made list of such uses; let the children discover many of them for themselves. The teacher will of course at times lend a helping though unseen hand. The children have long been graded on the basis of 100; so many per cent of milk is butter fat; so many per cent of whole wheat is used in making white flour; so many per cent of certain armies was killed in the World War, and so many per cent was wounded. In certain mines so many per cent of the ore is metal, etc. When children once get started finding such uses of percentage, they will soon have enough examples to bring home to them, to some degree at least, the tremendous part which percentage has come to play in modern life.

The real advantage in the use of per cents lies in the fact that numbers reduced to per cents are in a convenient form for comparison. A farmer takes the milk of two cows to the creamery and is told that one contains 3.8% butter fat and the other 4.5%; the comparison is then obvious. Suppose he had been told that one sample of milk contained 19/500 and the other 9/200 of butter fat! A miner has ore tested and is told that it contains so many per cent metal. We say that 5.8% of the population in a certain state is illiterate while in another state only 1.3% is illiterate. In reality it is not easy to overestimate the importance of the use of percentage, simple as it is.

180. The base, rate, and percentage. — In every problem in percentage two of the three elements, *base*, *rate*, and *percentage* are given, and the problem is to find the one remaining. Even in a clearly stated problem there may be some difficulty in deciding which two of these three are given. For this reason problems in percentage should be read just for the purpose of identifying the given elements and deciding which one is to be found. By turning the attention solely to this problem without obscuring it by a mass of computation, the child will become more effective in planning the solution of the problems. Percentage is of sufficient importance to justify considerable attention to this phase of it.

181. The chief difficulty in teaching percentage. — This difficulty consists in fully recognizing that there is no new mathematical element in percentage, in directing the instruction in strict accord with this fact, and at the same time developing in the pupil the permanent habit of thinking directly in terms of per cents. In our everyday lives we do not regard per cents as decimals or as fractions but as primary facts with which we work and think. At a moment's notice we are of course ready to say that a given per cent is so many hundredths, but that is not in our consciousness when we are making practical use of per cents. Drills, such as those suggested in section 178, will help to develop the habit of thinking directly in terms of per cent; comparison of related numbers reduced to per cents will serve admirably. Practical use in later life is sure to develop this habit in the case of any fairly intelligent person. We should do what we can to develop it in the schools.

REVIEW QUESTIONS

1. Discuss fully the relation of percentage to the preceding topics in arithmetic. What new elements are introduced into arithmetic in the subject of percentage? (§ 177.)

2. What types of elementary drill may be used in teaching percentage? (§ 178.)

3. Give numerous examples of the practical uses of percentage. How may the pupils be led to make lists of such uses? What is the chief advantage in using per cents? (§ 179.)

4. How may the meanings of "base," "rate," and "percentage" be taught so that the pupils may acquire a real working knowledge of them? (§ 180.)

5. What is the chief difficulty in teaching percentage? (§ 181.)

6. Comment on the statement: "The subject of percentage is almost wholly a language lesson."

CHAPTER XII

THE SOLUTION OF PROBLEMS

182. Organizing the solution of problems. — It is trite to remark that all work in arithmetic may be divided into two parts, one of which consists in performing the four fundamental operations and the other in deciding how these operations are combined for the purpose of solving problems. The latter is the distinctive element in the solution of problems and will be referred to here as the "solution of problems." The methods generally in vogue, as revealed in current texts, are characterized by the absence of any effective development of general principles that underlie the solution of wide ranges of problems.

A public confession may not be out of place: Some fifteen years ago the writer made a collection of arithmetic texts in connection with his work as a teacher in the Summer School of the Chautauqua Institution at Chautauqua, N. Y. A careful study of more than a score of series of arithmetics revealed a surprising absence of any attempt to show that the solution of a large number of different groups of problems depends upon a single idea. Instead, each group of problems was studied from the "bottom up," with little reference to groups of essentially similar problems which had already been solved. There was scant attempt to develop far-reaching ideas. In a certain sense the child studying these books would *know* little more about the solution of problems when reaching the seventh grade than he did in the fourth.[1] It was this situation which attracted the writer's attention to the subject of elementary arithmetic. He has studied with diligence all the texts that have appeared since that time, and with one or possibly two exceptions the comment just made applies also to these later texts. The writer believes that it is possible to organize the ideas that underlie the solution of problems in such manner as to greatly unify and simplify the whole subject. Such organization would be characteristic not of any particular year's work but of the whole course. The work of the lower grades would contain certain forward-looking elements which the teacher could handle effectively only when fully aware of the purpose and plan of the work in all the grades. This is one of the reasons why the present book deals with arithmetic as a whole and not with each year's work separately. We now proceed to consider some of the elements that would seem essential in organizing the solution of problems.

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[1] The exact meaning of these statements will appear more clearly as we proceed through this chapter.

183. Difficulties in the simplest one-step problems. — The foundation of all solving of problems is the solution of one-step problems. The questions to be answered by the child are: What situations or relations require addition? What situations require subtraction? etc. Usually when the child meets such problems for the first time, they are separated into groups, such that all problems in one group require the same operation. After having studied formal work in addition, he will come upon a group of concrete problems all of which require addition. At this point the child is likely to show greater ability than he really possesses. Just because he has been studying addition, it is natural for him to add the numbers he finds in his problem. If problems requiring subtraction were presented to him at this time, we should find him attempting to solve them by adding. Similar sets of problems, all requiring subtraction, will be encountered after the formal study of subtraction, and so on for multiplication and division.

At some point in the third grade, groups of problems will be encountered, some of which require addition and some subtraction (practically all one-step problems). The child's performance will now be a disappointment both to himself and to the teacher. The fact is that the child is now encountering something more difficult than the performance of the fundamental operation which he has learned; for the first time he has really encountered the difficulty of "solving problems."

It is believed that not sufficient attention has been given to the questions: "What kind of relations require addition?" "What kind, subtraction?" etc. Frequently difficulties persist because they are not clearly recognized and dealt with thoroughly at an early stage. It is my belief that no amount of so-called *analysis* will overcome the difficulty. It will be found that the analysis can be made only *after* the child has decided how to solve the problem. The fact is that the child usually can solve certain simple problems while he is unable to solve others involving exactly the same principle. The following series of examples will illustrate:

1. If you read 4 pages yesterday and 3 pages to-day, how many did you read in all? How do you find out?
2. We walked 12 blocks in the afternoon and 9 blocks in the forenoon. How many blocks did we walk altogether? How do you find out?
3. We drove our machine 78 miles this forenoon and 94 miles this afternoon. How many miles did we drive to-day? How do you find out?
4. A woman paid $1.25 for meat, $2.58 for potatoes, $.85 for butter, and $1.75 for groceries. How much did she spend in all? How do you find out?

These problems all involve addition. They were presented to third grade pupils who had learned to add quite effectively. Their equipment for these problems so far as the fundamental operations are concerned was entirely adequate. The results were as follows: All members of the class solved the first problem correctly and quickly. In answer to the question, "How do you find out?", they said, "4 and 3 are 7."

The majority solved the second problem correctly and answered the question, "How do you find out?", by saying, "12 and 9 are 21." Some who failed or hesitated said they could not tell how much 12 and 9 were. (Remember these children could add large numbers quite effectively.) None of the children solved the third problem promptly. They said they could not tell how much 78 and 96 were. (No suggestions or hints were given; no questions by the children were answered.) Finally a few obtained the correct answer. After a considerable time none had solved the last problem. The probable reason for this failure, which on the face of it would seem impossible, is given in the next section.

184. Organizing the solution of the simplest one-step problems. — These children were then put on a systematic study of one-step problems. The study began with problems, such as No. 1 in the preceding section. The child was led to say that he solved it by *adding* 4 and 3 — not merely by saying that "4 and 3 are 7." The idea was developed that *and-problems* were solved by *adding*. The children then made up "and-problems" and said that they solved them by *adding*. After this was done, problems 3 and 4 of the preceding section presented no difficulty. They then said the answers were found by *adding*, and the work of adding was done promptly and accurately.

What was lacking in the minds of these children was an *effective* connection between the *and-relation* which they recognized clearly in each problem and the formal process which they had learned under the caption "addition." It would be easy enough to tell the child that "*and-relations* require *addition*," and it would not be difficult to get him to repeat it. But too often we deceive ourselves by supposing that words really represent ideas. The "sounding brass" and the "tinkling cymbal" are not altogether of the past nor are they confined to theology. The essential point is that the child should be caused to *work* with the *and-idea*, which is already clear to him, and to learn by actual experience how it carries with it the operation of addition which with him is a recent product of his school life. What is needed is to bring the old familiar idea into the center of consciousness and to make it remain there while the other ideas which are to be connected with it are brought in and adjusted to it.

The final step in this study was to get the children to formulate "problems without numbers" corresponding to the concrete problems studied. Such problems were:

1. If you know how many pages you read yesterday and how many you read to-day, how do you find how many you read altogether?
2. and 3. If you know how far you went in the forenoon and also how far you went in the afternoon, how do you find how far you went altogether?
4. If you know how much you spent for this thing, for that thing, and for the other thing, how do you find how much you spent for all of these?

These problems were made by the children under fairly well-concealed stimulus and prompting. The answers were easy for all of them. Soon the children began to make more complicated problems, as, "If you know how many pages you read Monday, how many Tuesday, how many Wednesday, etc., how do you find how many you read the whole week?" By this time these children knew much more about what kind of problems require addition than at the outset. But let us not be too sanguine. When they came to a set of miscellaneous problems, they still had some trouble though no doubt much less than they would have had if the connection between *and* and *addition* had been slurred over.

Later a similar study was made of problems requiring subtraction. The following problems were used:

1. If you had 7 apples and ate 3, how many would you have left? How do you find out?
2. John is 8 years old now. In how many years will he be 11 years old?
3. Tom is 13 years old and his sister is 5 years younger; how old is she?
4. A man had $4.50 and spent $1.75; how much had he left?
5. Henry's grandfather is 67 years old and his father is 39 years younger; how old is Henry's father?
6. If you have $1.45 now, how much more must you get to have $6.50?

With each problem was the query, "How do you find out?"

Again problems 1, 2 and 3 caused no trouble, while 4, 5 and 6 caused trouble. The reason is now obvious and the remedy equally so. However, to decide which problems require subtraction is much more difficult than to decide which ones require addition.

"If one number is taken away from another, how much is left?"

"How much must be added to one number to make the sum another number?"

"What is the difference between two numbers?", or more specifically,

"How much greater is one number than another?"

"If we know the sum of two numbers and also one of them, how do we find the other number?"

All these appear to the child as fairly distinct kinds of problems, and all of them are solved by subtraction.

After the children had mastered multiplication, the following problems were used:

1. If writing tablets cost 8 cents apiece, how much would 4 tablets cost?
2. If children go to school five days a week, how many days do they go to school in 10 weeks?
3. If a child's school desk costs $6.75, how much would 6 such desks cost?
4. If a camping outfit costs $17.25, how much would 16 such outfits cost?

Again problems 1 and 2 caused no trouble, while 3 and 4 caused considerable trouble, though not so much as in the former cases. The trouble was not due to the dollar sign or the decimal point; the children had no trouble in performing the multiplications. The trouble was that they did not know that they were to *multiply*.

In all cases where trouble occurred, it was due to the same cause; the difficulty in all cases was connected with large numbers. Where the numbers were small enough to enable the children to do the work mentally and with ease, they had no trouble in deciding what to do; but they had very great trouble with exactly the same type of problems where the numbers were large enough to require written work — and that in spite of the fact that they were quite proficient in the kind of written work required.

On reflection one sees that this is precisely what should be expected. The child plays forward and backward over the lower part of the number system in a simple and direct fashion of his own. If the question is, "How many do 3 and 4 make?," the child starts, say with 4, and counts forward 3 steps arriving at 7. "Find a number 3 less than 7," and the child counts backward 3 steps from 7 arriving at 4. "How much must be added to 4 to make 7?," and the child manages to see that the interval between 4 and 7 is 3. "How much larger is 7 than 3?," and the answer is found in the same original and unsophisticated way. "What is the difference between 4 and 7?," and the child may hesitate until the question is changed to, "How much larger is 7 than 4?" or "How much less is 4 than 7?" When stated in this way, the child gets the result and probably also learns the real meaning of "difference." "What is 2 times 4?," and the child says, "two 4's is 8." The process with him is the same as the one he uses to find the sum of numbers though he may not recognize it as *addition*. Similarly 3 × 4 means 4 *and* 4 *and* 4, and the real process is again addition.

In answering all these questions the child never thinks of the *addition*, *subtraction*, or *multiplication* that he has been studying as formal processes. He moves about in the number system as one walks about in one's neighborhood without thinking of north, south, east, or west. Instead of being a source of wonder it is therefore to be entirely expected that a child may be able to answer such questions as those given above and at the same time be entirely unable to decide which one of the several formal processes to use in a particular problem. Indeed it may well be that it does not occur to him that one of these processes has anything to do with the problem in hand. Sailing by compass is a very different matter from following well-known paths to which the feet of one's childhood were accustomed.

It is only as the teacher appreciates the large gulf that exists here that proper instruction can be given. In this particular matter the textbook can only point the way. The oft-repeated drills, the many simple questions, the constant watching of the growing understanding — these must be very largely the work of the teacher.[1]

185. One-step problems reduced to two classes. — The four fundamental operations serve to divide all one-step problems into four classes; namely, those solvable by the four operations respectively. Since subtraction is the inverse of addition and division the inverse of multiplication, we may divide all one-step problems into two classes, one class consisting of those solvable by means of addition or subtraction and the other class consisting of those solvable by means of multiplication or division.

The equality, $3 + 4 = 7$, may be made into a problem by striking out any one of the three numbers involved. If the 7 is stricken out, we have $3 + 4 = ?$, which is a problem in addition; if 3 or 4 is stricken out we have $? + 4 = 7$ or $3 + ? = 7$, either of which is a problem in subtraction. Any problem of our first class comes under one of these forms.

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[1] The makers of texts are hard enough put to it to exclude matter so as to make the book come within the limits set by the publishers. The latter know only too well that a book fifty pages longer than the ordinary will carry the suggestion that it does not meet the modern requirements for simplicity and saving of time. Publishers can not publish books unless there is a prospect that they will sell, and only those books which are accepted by the publishers see the light.

The equality, $3 \times 4 = 12$, may likewise be made into three problems by striking out any one of the three numbers involved. If the 12 is stricken out, we have $3 \times 4 = ?$, which is a problem in multiplication; if the 3 or 4 is stricken out, we have $? \times 4 = 12$ or $3 \times ? = 12$, either of which is a problem in division. Every problem of our second class comes under one of these forms.

For the vast majority of teachers it is needless to remark that the classification given above should not be stated to the child in words at the outset; the material of the classification should be built up gradually, *culminating* in a final verbal *summary*. This summary may not come before late in the fourth grade or possibly in the fifth, but the teacher should be conscious of it all along. She must know the course of the whole journey, for this influences the direction of the steps taken each day.

186. The addition-subtraction group of problems. — In every problem of this group the idea of the *sum* of two numbers is present. The first step is to decide that the sum of two numbers and not their product is involved. All *and-relations*, all relations represented by "how much greater than?" or "how much less than?" as well as certain others of a type illustrated below involve a *sum*. The next step is to decide which number is the sum. (In every problem two of the three numbers involved will be given, either explicitly or implicitly.) If the sum is not given, the two numbers that are given must be added. If the sum and one of the addends is given, the problem is solved by subtraction, the given sum being the minuend and the other given number the subtrahend.

Let us illustrate: "There are 35,639 people in a county and 17,844 live in the city. How many live outside the city?" A child in the third or even the fourth grade may readily have trouble with this problem. If the solving of addition-subtraction problems has been properly developed and *built into the child's intellectual anatomy*, he might think of this problem something like this: The sum of those that live in the city and in the country is 35,639 and one of the numbers, those who live in the city, is 17,844. Hence the number of those who live in the country is found by subtracting 17,844 from 35,639. This whole idea would come very rapidly and possibly some of the details would fail to appear in consciousness. *A sum, one of the addends, subtract* — these ideas would flash through his mind in less time than it would take him to write down the first number.

In some problems leading to subtraction it may be more natural to think directly of the "difference" of two numbers, in which case this should be done. However, if the relations leading to subtraction are to be thought of, either *all* as involving "differences," or *all* as involving "sums," it will be found that the "sum" idea is at least as natural and easy as the "difference" idea. It arises directly in a larger number of practical cases.

A few more special problems of the addition-subtraction class may be mentioned. In treating the subject of interest, *amount* is defined as the sum of the principal and the interest.[1] The moment this definition is given any one of the three numbers involved may be found when the other two are given. In some cases the subtraction form is given directly in the definitions.

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[1] There is some objection to this use of the word "amount" as there is to certain other terms in common use in this subject. Thus: "As to the use of terms like 'base,' 'rate,' 'percentage,' 'amount,' and 'difference,' there is but little that can be said in their favor. They were invented in the rule stage of arithmetic and have served their purpose." Smith, *op. cit.* (p. 111), p. 183.

The fact is, however, that whether or not we use "rules" for solving problems in interest the objects represented by these words exist, and we sometimes need to refer to them. It would seem a little drastic to scrap these terms along with the "cases" in percentage. Their chief offense would seem to be that they have been associated with certain elements the abandonment of which is being urged.

Thus: *selling price* – *buying price* = *gain*
buying price – *selling price* = *loss*.

Aside from the simple problems of the early grades the problems in the addition-subtraction class are much less diversified in kind and less important so far as learning to solve problems is concerned than are those in the multiplication-division class.

187. Early preparation for the addition-subtraction problems. — The so-called addition method (Austrian method) of subtraction forms a natural basis for the solution of this class of problems. It serves to point out constantly that subtraction is the inverse of addition, that it consists in finding what number must be added to a given number to make the sum another given number. If subtraction is approached from this point of view, there will be much early drill like the following: Find the missing numbers in:

$3 + 4 = ?$
$? + 4 = 7$
$3 + ? = 7$

$9 + 7 = ?$
$? + 7 = 16$
$9 + ? = 16$

$8 + 5 = ?$
$? + 5 = 13$
$8 + ? = 13$

$6 + 9 = ?$
$? + 9 = 15$
$6 + ? = 15$

On proper occasion it is pointed out that finding the missing number in $? + 4 = 7$ is *subtraction*. Later when the child comes to formal subtraction, he has for example:

SUBTRACT: 874 He says: $4 = 2 + 2$. Write 2.
352 $7 = 5 + 2$. Write 2.
522 $8 = 3 + 5$. Write 5.

In this way the definition of subtraction as the inverse of addition is built solidly into the child's habits of work. This then forms a natural basis for the treatment described in the preceding section of problems requiring subtraction.

188. The multiplication-division group of problems. — Very many of the numbers with which we have to deal in solving problems in arithmetic are connected by the relation that one of them is the product of two or more of the others. The list is truly formidable. To enumerate the more important cases:

Area of rectangle = length × width.
Volume of solid = length × width × depth.
Distance = speed × time.
Cost = price × number of articles.
Interest = principal × rate × time.
Percentage = base × rate.
Discount = list (marking) price × rate of discount.
Commission = amount involved × rate of commission.
Gain = cost × rate of gain.
Loss = cost × rate of loss.

Or in some cases:

Gain = selling price × rate of gain.
Loss = selling price × rate of loss.

This list could be extended much further. The solution of the multitudes of problems in this class will be considered in part in this chapter after we have considered the early basis for the solution of such problems and in part in the various chapters devoted to such special topics as interest, loss and gain, etc. We now turn to the consideration of the early basis for the solution of such problems.

189. Early basis for the solution of multiplication-division problems. — A combination, such as $3 \times 4 = 12$, contains the key to the solution of all problems of the multiplication-division class. Whether or not the child learns to use this key depends upon the definition of division which is developed in the early part of the course. If the idea of division is developed as something apart from multiplication, all is lost. In many schools, there is much talk of "How many times is 3 *contained* in 12?" and little emphasis on the fact that $12 \div 3 = 4$, *because* $3 \times 4 = 12$.

In these schools division is defined as "the process of finding how many times one number is contained in another." Division is said to answer the question, "How many times may one number be subtracted from another?"

From the logical side obvious objections may be urged against such definitions. How many times may 2 be subtracted from 7? Evidently 3 times; but $7 \div 2$ is not equal to 3. How can such definitions be applied to $\frac{1}{3} \div \frac{1}{2}$? To say that $\frac{1}{2}$ is contained $\frac{2}{3}$ times in $\frac{1}{3}$ reminds one of the man who tried to look out of the window $3\frac{1}{2}$ times. Again $\frac{1}{2}$ can not be subtracted from $\frac{1}{3}$ (at least, not in arithmetic), and hence $\frac{1}{3} \div \frac{1}{2}$ would have no quotient.

But after all these are matters to which we might adjust ourselves. There is a widespread notion that human beings are by nature logical and self-consistent. We are nothing of the sort; we can all be illogical and inconsistent when it serves our purpose. The point here is that the definitions given above do not serve our purpose, and in that case the fact that they are logically unsound becomes a grave matter. "Officially," therefore, we will discard these definitions on the ground of their logical shortcomings. At any rate, from the writer's point of view they should be discarded, whatever may be the most available respectable reason.

Throughout all mathematics, except in some cases in elementary arithmetic, division is defined as the inverse of multiplication. One element of elegance and smoothness in this definition appears when it is placed side by side with the corresponding definition of subtraction:

Definition of Subtraction	Definition of Division
Subtraction is the process of finding one of two numbers when their sum and the other number are given.	*Division is the process of finding one of two numbers when their product and the other number are given.*

If this definition is used, then $12 \div 3 = 4$, *because* $3 \times 4 = 12$; and $\frac{1}{3} \div \frac{1}{2} = \frac{2}{3}$, *because* $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$. It is certain that no objections can be urged against this definition on logical grounds. Moreover, it seems to serve the purposes of arithmetic admirably. If, in any one of the equalities on page 305, all but one of the numbers involved are given, this one can now be found either by multiplication, in which case the operation to be used is explicitly given by the equation, or by division, in which case the operation to be used follows directly from the definition of division.

Suppose we know that in a rectangle:

$$\textit{area} = \textit{length} \times \textit{width}.$$

If the area and either the length or the width are given, we shall then know *without further thought* that the missing number may be found by dividing. If the child has trouble, we shall only need to write $3 \times 4 = 12$ on the board, strike out the 3 or the 4, and ask how it may be found. "Divide 12 by 3," (or by 4) says the child. Very well! If in "area = length × width" say length is stricken out, how may it be found? "Divide the area by the width," is the answer. If misgivings arise here as to the proper treatment of abstract and concrete numbers, a reading of section 194 will serve at least to indicate the writer's attitude on this matter.

If this point of view is adopted, there will be much work like the following in the early grades, say in the third grade:

Find the missing numbers in:

$7 \times 6 = ?$	$5 \times 9 = ?$	$8 \times 7 = ?$	$6 \times 9 = ?$
$? \times 6 = 42$	$? \times 9 = 45$	$? \times 7 = 56$	$? \times 9 = 63$
$7 \times ? = 42$	$5 \times ? = 45$	$8 \times ? = 56$	$6 \times ? = 63$

The following forms will often be used interchangeably:

$56 \div 8 = ?$	$42 \div 6 = ?$	$56 \div 7 = ?$	$63 \div 9 = ?$
$\begin{array}{r} ? \\ 8\overline{)56} \end{array}$	$\begin{array}{r} ? \\ 6\overline{)42} \end{array}$	$\begin{array}{r} ? \\ 7\overline{)56} \end{array}$	$\begin{array}{r} ? \\ 9\overline{)63} \end{array}$

In each case, 56 ÷ 8 = 7, *because* 7 × 8 = 56. That is, the definition of division is *built up* as the inverse of multiplication. It thus comes to pass that success or failure in solving problems in the later grades depends in no small degree upon the type of work done in the second, third, and fourth grades.

The general idea discussed in section 184 must not be forgotten. It is perfectly possible that the child may know (at least say) that the missing number in 7 × ? = 56 is obtained by a process called "division" and still be unable to find the missing number in 456 × ? = 137498, even though he knows how to perform the process of long division. If, however, some of the problems in long division are stated in this form, he will see the point very easily, and there will be no further trouble. Many of the difficulties of arithmetic are like making Columbus' egg stand on end: no one thought of chucking it down a bit. Nor are we justified in our not infrequent "amazement at the child's stupidity in not seeing the obvious." How many instances there are in the history of science where it seems almost impossible that brilliant people, such as the Greeks, for instance, should have escaped making important steps in advance, which to us seem absolutely obvious! The use of the numeral 0 (zero) is a good example. But it is said: Does it follow that we should leave nothing to be discovered by the child for himself? No. What to leave and what not to leave? Oh, therein lies a good part of the art of teaching.

190. Actual solution of multiplication-division problems. — The substance of this section has been suggested in the one that precedes. To make the general point of view clear, we will consider the solution of two problems.

PROBLEM 1: The area of a rectangle is 48 square feet and its width is 6 feet. Find its length.

We give first what, it is believed, may justly be called the usual schoolroom method of solving and explaining this problem:

If the rectangle is 6 feet wide, then along one end of it there is a row of 6 square feet. Since there are 48 square feet in the whole rectangle, and since 6 is contained 8 times in 48, there are 8 such rows. Hence the rectangle is 8 feet long. But finding how many times 6 is contained in 48 is division; hence the length is found by dividing the number of square feet by the number of feet in the width.

This explanation would of course be accompanied by appropriate drawings.

The type of solution proposed here is as follows:

The rule for finding the area of a rectangle has already been fully developed and used entirely independently of problems of this kind. For this purpose drawings and other means have been used, and the rule has been stated in the final form:

$$\textit{area} = \textit{length} \times \textit{width}.$$

This work would have to be done no matter how the present problem is solved.

This information being already at hand, the present problem is solved directly by referring to the problem of finding the missing number in $6 \times ? = 48$. But this problem the child has solved repeatedly in connection with learning the multiplication table and has learned to regard it as a problem in division.

It should be clear that the work of developing the method for finding the area when the length and the width are given is precisely the same whether the one or the other of these two methods is used for solving the present problem. On this point, see, however, section 193.

PROBLEM 2: One year the net income of a corporation was $1,242,296.75 and the next year the net income was $1,476,846.50. How many per cent did the net income increase?

Whatever method is used it is necessary to understand that the first step is to find the increase of the net earnings the second year over those of the first year. That is, $1,476,846.50 –

$1,242,296.75 = $234,549.75. The child must also understand that the next step is to find how many per cent $234,549.75 is of $1,242,296.75. That is, he must identify in the problem the elements "base," "rate," and "percentage."

At this point the usual analysis is:

> 1% of $1,242,296.75 is $12,422.9675.
> $12,422.9675 is contained ($234,549.75 ÷ $12,422.9675 =) 10.82 times in $234,549.75.
> Hence the increase in net income is 10.82%.

The solution proposed here is to note that:

percentage = base × rate.

Hence by direct use of the definition of division or, putting it concretely, by reference to 4 × ? = 12, we know that the rate is found by dividing the percentage by the base ($234,549.75 by $1,242,296.75) and writing the result as per cent (by moving the decimal point two places toward the right).

191. List of problems solved by direct application of the definition of division. — We now assume that the child knows the definition of division given on page 306 and knows it not in words only, but as a living, functioning truth. If he also knows that:

area of rectangle = length × width,

then he will know *at once* how to find any one of these three numbers if the other two are given.

If he knows that:

volume of rectangular solid = length × width × depth,

he will know *at once* how to find any one of these four numbers when the other three are given. If there should be a slight difficulty because the product of three instead of two numbers is involved study the examples:

$$24 = ? \times 3 \times 4, \quad 24 = 2 \times ? \times 4, \quad 24 = 2 \times 3 \times ?$$

and state the rule for finding the missing numbers in them. Similarly, if the child knows the equations given on page 305, he will know *at once* how to find area, length, or width of a rectangle; volume, length, width, or depth of a rectangular solid; distance, speed, or time in velocity-distance problems; cost, price, or number of articles bought or sold; interest, principal, rate, or time in problems in interest; percentage, base, or rate in problems in percentage; discount, marking (or list) price, or rate discount in problems in discount; commission, amount involved in transaction, or rate commission in problems in commission; gain, cost (or selling price), or rate loss or gain in problems in loss and gain when the necessary data are furnished. That is, he knows *at once* how to solve all problems in subjects which now occupy about two years of time in the schools.[1] The list is truly formidable and could be extended considerably. Problems in taxation, stocks and bonds, investments, etc. come under this same head.

192. The value of far-reaching methods. — It is not easy to overstate the value of using the same underlying idea in the solution of a wide range of problems. The idea itself is clarified every time it is used, and its use becomes easier and more effective. The detailed analysis does not necessarily bring out the elements of similarity in different problems; it often brings out elements of diversity. In any event the similarity is not insisted upon and the result is that each kind of problems is solved in a way essentially peculiar to itself. The child is led to think of each kind of problems, not as related to a host of other kinds, but as a thing apart to be mastered separately — and soon to be forgotten separately. If the child uses the method proposed here, he *can not* forget how to find the base or the rate so long as he knows the fundamental relation,

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[1] For definition of "solution of problems," see section 182. For further discussion of the solution of problems enumerated, see Chapters XIII–XVI.

$$\textit{percentage} = \textit{base} \times \textit{rate},$$

and, if he does not know this relation (or its equivalent), he can not solve *any* problem involving percentage, no matter what method he uses.

By making a thorough study of the relations enumerated on page 305 and consistently using the method proposed here there is little doubt that the time required to master the problems of finding the quantities enumerated on page 311 would be reduced by at least one half when compared with the time required by the analysis method or by the method of separate "cases" and "rules." Moreover, he will be far less likely to forget them than when learning by either of the other methods.

Objections that will be urged against this method are obvious: It can easily be shown that the mastery of the problem of finding the length of a rectangle as given on page 309 may be achieved in a shorter space of time by the analysis method than by the method proposed here provided we charge against our method the time required to develop the preliminary machinery. Now, that would seem to settle the matter; but it does not. We are not considering the time it takes to learn to solve *one* problem but to solve a large number of, at least apparently, different kinds of problems. The saving by using our method comes from the fact that the same machinery serves for large groups of problems. An example from a different field may be illuminating: When the Indian wanted a pole for his tepee, he cut it down, tied one end to his saddle, and dragged it home. That was his way of hauling lumber. It was a direct and common-sense way. We white folks do it differently. When we want to haul lumber we build roads, we mine ore and convert it into steel rails and freight cars and locomotives, we build wagons and sleds. But when we are once ready, how we do haul lumber! The analysis method or the "case" and "rule" method may be called the Indian method — a sort of hand-to-mouth method. It does have an appeal if we do not look far ahead. The teacher in the fourth grade, teaching a child to find the length of a rectangle by the analysis (Indian) method, does not prepare as she might for the work in the later grades. It is only as we organize far-reaching methods with reference to the whole school course and with reference to the whole of life that we shall get away from the hand-to-mouth method — the method characteristic of the life of the savage. The savage does not do things with a far-reaching purpose; he lives the life of to-day, "taking (little or) no thought for the morrow." That is the chief reason why he is disappearing from those parts of the earth where farsighted people care to make their habitation.

193. Problems without numbers. — Several problems without numbers are given on page 298. The answer to the question raised in such a problem is a rule for solving all problems of the kind represented by the problem in question. Thus the answer to the question, "If you know howmany pages you read yesterday and also how many you read to-day, how do you find how many you read on the two days?," tells how to solve this problem no matter how many pages were read. The answer to the question, "How do you find the area of a rectangle when its length and width are known?," is a rule for finding the areas of all rectangles.

A problem without numbers may always be regarded as a summary of a group of related problems. In some cases the problems represent a very narrow simple group, such as the first problem given above. It is limited to the number of pages read on two consecutive days. In other cases the problems represent a much more general group, such as the second problem given above, which includes all rectangles. The relations given on page 305 are answers to problems without numbers, each of which represents a large body of problems.

In the early part of the course each problem or each group of strictly similar problems may be followed by a problem without numbers. Such problems may be made up on the spot, either by the teacher or the pupils. From time to time a group of such problems may be given as a review of the problem work that precedes. Some of the newer texts contain pages of such problems scattered throughout the series.

In case problems without numbers represent one-step problems they require simply the recognition of relations that lead to addition, subtraction, multiplication, or division. Such problems have been considered in detail in this chapter. In case they represent problems involving more than one step, the solution consists of a statement of the consecutive steps and the operation required in each step. This is considered further in section 196 which deals with analysis.

The value of problems without numbers lies in the fact that they invite the mind to dwell solely on the questions as to *how* such a problem is to be solved without the distraction caused by performing the computation. They are strictly analogous to the so-called literal problems in algebra, where the solution is a formula showing what operations are to be performed. In arithmetic the answer gives the same information in the form of a verbal statement or rule.

194. Operations on concrete and abstract numbers. — Certain rules have been laid down in regard to operations on concrete and abstract numbers. For the purpose of these rules abstract numbers are regarded as being of the same kind. The rules are:

1. Addends must be numbers of the same kind, and the sum is a number of the same kind as the addends.
2. Minuend and subtrahend must be numbers of the same kind, and the remainder is of the same kind as these.
3. The multiplicand may be an abstract or a concrete number; the multiplier must always be abstract, and the product is always of the same kind as the multiplicand.
4. The dividend may be abstract or concrete. If the dividend is abstract, the divisor and quotient are both abstract. If the dividend is concrete, the divisor may be a concrete number of the same kind, in which case the quotient is abstract; or it may be an abstract number, in which case the quotient is a concrete number of the same kind as the dividend.

All (or nearly all) agree that these rules are correct, though there is no such unanimity as to their usefulness. It is asserted that it is illogical, for instance, to multiply dollars by days, though it is not so clear as to *why* it is illogical; and it is urged that we must never multiply 16 feet by 18 feet, not even to obtain the area of a rectangle. This forms the basis for rules 3 and 4 (rule 4 follows from 3 because division is the inverse of multiplication).

It is urged in support of the usefulness of these rules that by carrying them out faithfully the kind of number represented by the result will be made to appear as a direct logical necessity. It is asserted that by referring to these rules one may learn whether a result represents feet, or dollars, or days, or tons, or an abstract number. The practical difficulties of carrying out the program implied here are enormous. In fact no text and no teacher carries them through to the bitter end; a compromise is always made somewhere. The most obvious remark is that we do not need these rules to determine the character of the results. In most cases a result of a certain kind is required by the terms of the problem.

At $4.00 a day how many *dollars* does a man earn in 15 days?

How many *square feet* are there in a rectangle 84 feet long and 36 feet wide?

If a man builds 25 rods of fence in one day, how long (how many *days*) will it take him to build 200 rods of fence?

In the first example the number of dollars, considered as a concrete number, is multiplied by the number of days, considered as an abstract number, and the result is dollars. Why do we not consider the number of dollars as abstract and the number of days as concrete? Because this would give a result in terms of days which is not what we want.

In the second problem the number of feet in the length and also in the width are both concrete numbers. To satisfy the rule and to get the answer wanted, we regard both of them as abstract and then multiply the product of these by 1 square foot.

In the third problem number of rods is divided by number of rods and somehow the result turns out to be days. The process is a little confusing here, but at all events we know the result must be *days;* the problem asks for *days*. If we were to take some pains we should probably be able to arrange a procedure which would give the result in days and also make proper use of the rules. The simple truth is that these rules serve well enough as guides provided we know the way; otherwise they are of little value.

In actual practice all numbers used in the operations are regarded as abstract and the denominations of the results are determined, as they always must be determined, from considerations entirely apart from the rules stated on page 315. Not only are these rules useless for the purpose for which they are mainly urged but, if followed consistently, lead to unnecessarily awkward computations or else make special explanations necessary.

At $6.00 a day how much will 358 men earn in one day?

Knowing as we do that the result must be dollars we shall be obliged to multiply $6.00 by 358. That is, 358 must be the multiplier. The operation is shown at the right. No one outside the schoolroom solves this problem this way; 6 would invariably be used as the multiplier. To avoid this difficulty, it is explained that 358 × $6 = 6 × $358. This makes it possible to use 6 as the multiplier and still save the rule. In all business practice the operation would be performed on abstract numbers and the dollar sign prefixed to the result, *because* a result in terms of dollars is required. The solution would likely be written down on a line as follows:

358 days work at $6 per day $2148,

the computation, unless carried out mentally, being recorded on a separate scrap of paper, which is thrown away.

$$\begin{array}{r} \$6 \\ \underline{358} \\ 48 \\ 30 \\ \underline{18} \\ 2148 \end{array}$$

One point that has been especially emphasized is that both terms of a ratio must be numbers of the same kind. We may speak of the ratio of the number of men in two armies in the late war, but we must not speak of the ratio of the number of *men* to the number of *women* in the state of Montana, or the number of *Japanese* to the number of *Chinese* in California, or the number of *negroes* to the number of *whites* in Alabama, or the number of *Jews* to the number of *Gentiles* in New York City — so at least say the authorities on school arithmetic, though statisticians of high and low degrees are constantly using such ratios. In some cases we have been too "logical" to be sensible.

The following examples quoted by Thorndike[1] from Horn's thesis are instructive: "A seventh grade girl gets the population of the United States in 1820:

7,862,166	whites
233,634	free negroes
1,538,022	slaves
9,633,822	

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[1] *Op cit.* (p. 42).

"Many classes in the Eastern Kentucky State Normal have been given this problem to solve, — and they invariably get the same result:

"'In a garden on the Summit are as many cabbage heads as the total number of ladies and gentlemen in this class. How many cabbage heads in the garden?' And the blackboard solution looks like this each time:

29 ladies
$\underline{15}$ gentlemen
44 cabbage heads"

The point we wish to make here is not that the last result is necessarily wrong but that it illustrates a practice which is not always safe.

195. Solving one-step problems by analysis. — In many cases ordinary one-step problems have been broken up into many steps by analysis. We give two examples:

Example 1: Two fifths of a number equals 12. Find the number.

Example 2: Goods are to be sold at \$28.50, and a discount of 5% must be given. What shall I mark the goods?

Solution of Example 1:[1]
(Usual analysis)

$^5/_5$ of the number = the number
$^2/_5$ of the number = 12
$^1/_5$ of the number = ½ of 12 = 6
$^5/_5$ of the number = 5 × 6 = 30

Solution of Example 2:[2]
(Usual analysis)

M. P. = $^{20}/_{20}$
Dis. = $^1/_{20}$ of M. P.
$^{19}/_{20}$ of M. P. = final S. P.
\$28.50 = final S. P.
$^{19}/_{20}$ of M. P. = \$28.50
$^1/_{20}$ of M. P. = $^1/_{19}$ of \$28.50
$^{20}/_{20}$ of M. P. = $^{20}/_{19}$ of \$28.50
= \$30 Ans.

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[1] Quoted verbatim from Brown and Coffman, *op. cit.* (p. 31), p. 49.

[2] Quoted in faithful detail from Klapper, *op. cit.* (p. 289), p. 288. The analysis is approved by Klapper, and an algebraic solution is given which is equally long. We gather from the context that M. P. represents "marking price," and S. P. represents "selling price," though no such explanation is given.

Example 1 is the example considered on page 318 and should be solved directly as a one-step problem.

$$12 \text{ is } {}^2/_5 \text{ of } 30, \textit{ because } 12 \div {}^2/_5 = 30$$

The problem simply requires us to find the missing number in ${}^2/_5 \times ? = 12$. Compare the discussion in section 189.

Example 2 is a simple one-step problem; namely, "find the missing number in $.95 \times ? = \$28.50$."

The missing number is therefore $\$28.50 \div .95 = \$30$.

These two examples serve admirably to show the simplicity of the method proposed here as against methods which have been and still are in very wide vogue in our schools. The large number of types of problems enumerated on page 311 are, in the majority of our schools, solved by methods analogous to the one shown on the preceding page on the right.

196. Analysis of problems. — Analysis means "taking apart." To analyze a problem is to take it apart so that each constituent element may be dealt with separately. Into how many parts shall a problem be divided? There are remote limits to the number of parts into which it *can* be divided. That is shown by the solutions given on page 318. The answer is that a problem which is not a one-step problem should be analyzed into parts each of which is a one-step problem.

Every problem requiring more than one step should be preceded (and in practically all texts is so preceded) by one-step problems of the types represented by the problems into which it is broken up by analysis. When these types of one-step problems have been mastered, no one-step problems should be analyzed. They are already in such form that we can handle them as wholes by one single operation. To break up such problems never makes for simplicity or for understanding; it causes nothing but trouble and confusion.

The analysis of problems containing more than one step is indicated in various forms in the written work required in the schools. The following examples will illustrate:

EXAMPLE 1: At $16.50 a ton what is the value of a load of hay weighing 2760 pounds?

SOLUTION: *Step 1.* 2760 ÷ 2000 = 1.38 (tons)
Step 2. 1.38 × $16.50 = $22.77 (value of hay)

EXAMPLE 2: An automobile is found to run 180 miles on 16 gallons of gasoline. At this rate how much does gasoline cost for a season's run of 7650 miles, if the price is 26 cents a gallon?

SOLUTION: *Step 1.* 180 ÷ 16 = 11.25 (miles per gallon)
Step 2. 7650 ÷ 11.25 = 680 (gallons)
Step 3. 680 × $.26 = $176.80 (required cost)

EXAMPLE 3: A farmer sold six loads of corn weighing 1870, 2040, 2090, 2300, 2620, and 2130 pounds respectively. At 64 cents a bushel, how much was this corn worth if one bushel of corn weighs 70 pounds?

SOLUTION: *Step 1.* 1870 + 2040 + 2090 + 2300 + 2620 + 2130 = 13050 (pounds)
Step 2. 13050 ÷ 70 = 186.4 (bushels)
Step 3. 186.4 × $.64 = $119.40 (required value)

Frequently the child is required to write a purely verbal comment on each step. This may be excellent training in language but can scarcely contribute to the understanding of the solution or to the teacher's knowledge of the child's attainments. If the child actually arranges the steps in the proper sequence, that is very good evidence of understanding; the number of *possible* arrangements of steps is so great that the correct sequence is very unlikely to come by mere chance. Of course, questioning may disclose that the child is dealing in property not his own. This, however, had best be investigated directly in the classroom, for requiring fuller written explanation is not likely to disclose the facts. It is as easy to borrow a fuller explanation as to borrow the more threadbare solution.

The above solutions are what may be called *direct solutions*. Some problems are solved more readily by *indirect solutions*.

EXAMPLE: If apples are sold for 2 for 5 cents, how much will one dozen apples cost?

SOLUTION: 12 is 6 times 2. Hence the price of 12 apples is 6 times the price of 2 apples, or 6×5 cents = 30 cents.

The steps in the direct solution would be: $5 \div 2 = 2\frac{1}{2}$ (cost per apple in cents). $12 \times 2\frac{1}{2}$ cents = 30 cents (cost of 12 apples). In simple problems which can be solved mentally, it is often best to devise indirect solutions.

We will now consider a special method of carrying out direct solutions.

197. Indicated solutions. — In many problems the work can be shortened by indicating the complete solution before performing any of the computations.

EXAMPLE 1: At $16.50 a ton what is the value of a load of hay weighing 2760 pounds (Example 1, page 320)?

SOLUTION: The number of tons may be indicated by $\frac{2760}{2000}$ and the value in dollars by $\frac{2760 \times 16.50}{2000}$.

This expression is written down directly, but in thought it involves the two steps shown on page 320.

EXAMPLE 2: An automobile is found to run 180 miles on 16 gallons of gasoline. At this rate how much does gasoline cost for a season's run of 7650 miles, if gasoline is 26 cents per gallon? (Example 2, page 320.)

SOLUTION: $\frac{180}{16}$ = number of miles per gallon; $7650 \div \frac{180}{16}$ or $\frac{7650 \times 16}{180}$ = number of gallons; and $\frac{7650 \times 16 \times .26}{180}$ is the required cost in dollars.

As in Example 1 the steps involved are exactly those given on page 320, but the expression $\frac{7650 \times 16 \times .26}{180}$ may be written down directly, the steps being taken in the mind as the expression is put down.

EXAMPLE 3: A farmer weighs the hay from a measured piece of land 12 rods by 18 rods and finds it to be $3^1/_5$ tons. At this rate what is the total yield in a field 60 by 95 rods?

SOLUTION: The area of the measured field is 18 × 12 square rods. Hence the yield on one square rod is $3^1/_5 \div (18 \times 12)$ or $\frac{16}{5} \div (18 \times 12) = \frac{16}{5 \times 18 \times 12}$. The area of the whole field is 60 × 95 square rods. Hence the total yield is

$$\frac{6}{5 \times 18 \times 12} \times 60 \times 95 = \frac{16 \times 60 \times 95}{5 \times 18 \times 12} \text{ (tons).}$$

Again the final expression is put down directly as the steps are taken one by one in thought. If the operations indicated by the steps were carried out separately, there would be a temptation to find the yield per acre which would increase the computation by two divisions by 160.

A correctly indicated solution is very good evidence that the child understands perfectly what steps are involved, and the remarks on page 320 in regard to further explanation apply also to this case.

There is a large class of problems like those just given whose solution may be indicated as a fraction whose numerator and denominator are the products of integers. The computation is carried out by means of cancellation where possible. In indicating such solutions the following rules on fractions are used:

To multiply a fraction by an integer multiply the numerator.
To divide a fraction by an integer multiply the denominator.
To multiply a fraction by a fraction multiply numerator by numerator and denominator by denominator.
To divide by a fraction invert its terms and multiply.

Indicated solutions have the advantage already noted in connection with "Problems without numbers," page 314, that the mind is left free to consider the steps involved without being distracted by intermittent work of computing.

198. Forms of solution. — It is important for many reasons that written work should be in proper form.[1] The solutions of problems should possess the following qualities:

(1) *There should be no incorrect statement.*
(2) *The work should show each step in the solution.*
(3) *The work should permit easy checking.*
(4) *The forms used should conform to business practice.*

Curiously enough there is not complete agreement as to whether certain forms are correct or incorrect. All will agree that it is not correct to say, "let $^5/_5$ = the number" (unless the number should happen to be 1). It is likewise incorrect to say, "let 112% equal the number," as it is to say that "a number has been decreased by 150%." Clearly no number can be reduced by more than 100%, though statements to this effect are frequently made not only by school children but by people of education and sometimes of distinction.[2] In adding mixed numbers children sometimes write $3\frac{1}{2} = {}^6/_{12}$, meaning of course $\frac{1}{2} = {}^6/_{12}$. In solving problems in *longitude and time*, pupils sometimes write 1 hr. = 15°. These are obvious errors which can be corrected easily.

We are not all agreed, however, whether or not it is correct to write

$$6 \text{ ft.} \times 12 \text{ ft.} = 72 \text{ square feet.}$$
$$3 \text{ ft.} \times 4 \text{ ft.} \times 10 \text{ ft.} = 120 \text{ cubic feet.}$$
$$16 \text{ ft.} \times \tfrac{3}{4} \text{ ft.} \times 2 \text{ in.} = 24 \text{ board feet.}$$

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[1] "David Eugene Smith says, 'It is the loose manner of writing out solutions, tolerated by many teachers, that gives rise to half the mistakes in reasoning which vitiate the pupil's work,' and 'teachers are coming to recognize that inaccuracies of statement tend to beget inaccuracy of thought and so should not be tolerated in the schoolroom.'" Brown and Coffman, *op. cit.* (p. 31), p. 44.

[2] Starch [*op. cit.* (p. 3), p. 189] quoting Proctor says, "The net increase in the total number of high marks has been 157%; the decrease in low failures, 188%." Such expressions are not uncommon either in speech or in print.

Educational authorities are strong against these forms[1] They deny, for instance, that multiplying feet by feet (if indeed such an operation is ever possible) gives square feet as a product; but on the other hand these forms are used freely (almost exclusively) by business men and engineers. The alternative forms usually recommended, 6 × 12 × 1 sq. ft. = 72 sq. ft.; 3 × 4 × 10 × 1 cu. ft. = 120 cu. ft.; 16 × ¾ × 2 × 1 board foot = 24 board feet, are never used outside the schoolroom. It requires courage to make recommendations in a case of this sort. The writer once taught in a rural school. Near by was a sawmill and lumberyard. Actual bills for lumber were obtained, copied, extended, and completely filled out. Inasmuch as no higher educational authority was at hand to set us right, we (teacher and pupils) used the ordinary business forms. We understood of course that the statements were elliptical. 16′ × ¾′ × 2″ was intended to give the dimensions of the plank. We knew further that the number of board feet was obtained by taking the product of two dimensions given in feet and one in inches. For the purpose of the multiplication the dimensions were of course regarded as abstract numbers. The equation, 16′ × ¾′ × 2″ = 24 board feet, meant simply that the plank described contained so many board feet and that the result was obtained by multiplying 16 by ¾ by 2. Similar stories may readily be told about 6 ft. × 12 ft. = 72 sq. ft. and 3 ft. × 4 ft. × 10 ft. = 120 cu. ft. My personal opinion is that in cases of this kind nothing is gained by overrefined logic and that the proper course is to learn to understand the situation and then use the current forms.

In the early parts of the course, while the children are still weak in the fundamental operations, the complete multiplications and divisions must of course be given, but later on they should *never* be given in written solutions of probems. See examples solved on page 320. Compare also page 321. These operations are never shown in business or technical papers. In practice the work of extending bills and other similar operations is never shown. The only concession to the needs of instruction that should be made is to have the *kind* of operations indicated. Thus while a business paper would say in effect:

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[1] "The following is an inaccurate expression: 4 ft. × 5 ft. = 20 sq. ft. . . . we may find its area by the following method: 4 × 5 × 1 sq. ft. = 20 sq. ft." Brown and Coffman, *op. cit.* (p. 31), pp. 47, 48.

$$314.6 \text{ is } 3.79\% \text{ of } 8294,$$

the school child's paper may read:

$$314.6 \div 8294 = .0379 = 3.79\%.$$

We shall not go far wrong in the matter of form if we adopt current business forms whenever possible, taking care that the real meaning of such forms is understood.

It is pretty generally agreed that the old program of carrying concrete numbers through a computation with their labels attached to them is in many cases very cumbersome and even silly. But the final result must be labeled. How are we to avoid writing an equation the first member of which is abstract and the second concrete? The writer has used the device of writing the label of the final number in parentheses just after the number, thus avoiding the formal difficulty and at the same time making a record of the thing represented by the formal result. See page 320.

Finally, while considering this matter of form, it should be said that the initiative of the child should not be withered by extremely high critical temperature. The writer has seen classes where there was such insistence on meticulous punctilio that the attention was almost entirely diverted from the real business in hand. The child must be allowed to *grow* into the ability to use perfect forms and perfect expression. "Normalcy" was not achieved in a day.

199. Crutches. — It is frequently the case that the form used in practical life is not the one that is most easily understood by the learner. For this reason special forms have been used at the outset for the purpose of facilitating initial understanding. When this is achieved, the change to the current forms is made. Such preliminary forms are called *crutches*. "Crutches" have been objected to on the ground that two forms used for the same purpose are confusing and that hence the final form should be used at the outset. On this ground objection has been made to the practice of first studying long division, using such examples as $6\,\overline{)\,497}$ which finally will be solved by short division. Again, when the borrowing method is used in subtraction, the figures in the minuend are sometimes changed to indicate the borrowing.

While the use of crutches should not be unduly encouraged, it is very likely that their danger has been overestimated. In only a very few cases is the child likely to insist on their use when he no longer needs them. Indeed, he will welcome the shorter form; the child is not so anxious to *understand* what he does that he will persist in using laborious forms. Suppose the example ${}^{3}/_{4} \div {}^{5}/_{6}$ is at first solved by multiplying both dividend and divisor by 24, giving $18 \div 20 = {}^{9}/_{10}$ as the result. When the child has learned to "invert the terms of the divisor," he will leave the first form quickly enough.

In those relatively rare cases where the child clings to the crutch a little persuasion will get him to leave it.

200. Checking results. — The results obtained in solving problems should be checked by showing that they satisfy the conditions of the problem. Such checks will prove that the proper method has been used and also that the operations have been carried out correctly. The same standard of accuracy should be maintained in solving problems as in the fundamental operations. The results must not only be correct; the computer must *know* they are correct. This can be ascertained by checking. See page 244.

There is no way however of checking the *interpretation* of the problem, for the same interpretation that was used in solving the problem will be used in making the check.

201. General suggestions on solving problems. — It will be helpful if each of the following steps is given separate and careful attention:

(1) *Read the problem with care.*
(2) *Plan the solution.*
(3) *Perform the computations required by the plan.*
(4) *Check the result.*

Of these suggestions the first is not the least important. Very frequently the inability to solve a problem is due to careless reading. If necessary the child may be asked to state what is given in the problem and what is required. One of the important elements in the study of arithmetic is, or at least should be, that it develops the habit of paying definite attention to each element in a statement.

The planning of a solution consists in selecting the proper operation in the case of a one-step problem and in making an analysis of the problem in case it can not be solved by one step. In making such plan it should be borne in mind that no one type of solutions is universally the best. In some cases an analysis like that given on page 320 may be the best type; in other cases it may be best to make an indicated solution of the type shown on page 321.

REVIEW QUESTIONS

1. Describe fully the difficulties that young children have in solving one-step problems. What seems to be the chief reasons for these difficulties? (§§ 183, 184.)

2. How may the difficulties just described be overcome?

3. Describe the addition-subtraction group of problems. How may a one-step problem be identified as belonging to this group? What preparation should be made early in the course for the solution of such problems? (§§ 186, 187.)

4. Enumerate problems which belong to the multiplication-division group. (§§ 188, 191.)

5. Describe the basis which should be laid early in the course for the solution of problems belonging to the multiplication-division group. Compare the definitions of subtraction and division used in this book. How do these definitions lead to the easy solution of many problems? (§ 189.)

6. Discuss the value of using a few far-reaching methods instead of many unrelated methods. (§ 192.)

7. Discuss the value of problems without numbers. Note that the answers to such problems consist of general rules each of which gives the method for solving whole classes of problems. (§ 193.)

8. Discuss fully operations on concrete and abstract numbers. What is the practical business usage in this respect? (§§ 194, 198.)

9. Contrast the solution of one-step problems by analysis, examples of which are given in section 195, with the direct solution proposed in this book.

10. What kind of problems should be analyzed and how far should the analysis be carried? Give examples. Also give examples of direct and indirect solutions. Under what circumstances should each of these methods be used? (§ 196.)

11. Give examples of indicated solutions. What are the advantages of indicating the complete solution before doing any of the computing? (§ 197.)

12. Discuss forms of solution. How do forms used in business practice differ from forms that have been used extensively in the schoolroom? (§ 198.)

13. Discuss the use of "crutches." What argument is used quite generally against crutches? Is this argument valid? (§ 199.)

14. How should results obtained in solving problems be checked? Is it possible to check the interpretation of a problem? (§ 200.)

CHAPTER XIII

DISCOUNT

202. Definition of discount. — Any article that has been offered or advertised for sale at a certain price and is then sold at a reduction from this price is said to be *sold at a discount*. As in all other applications of percentage no new mathematical principles are involved. The new elements involved in the study of discount are: the situation which gives rise to the problem, the business usages connected with it, and the language used in describing it.

In teaching this subject the problem is not to "define" discount in words but to bring home to the child the facts and the meanings of the terms used in such a way that they will be living realities. Let the children be asked to look through the papers for advertisements of goods offered for sale at a discount and to cut out such advertisements and bring them to school. If possible, catalogues should be obtained from business concerns, such as one of the great manufacturers of electrical appliances. In such a catalogue gross prices are given together with certain discounts from these prices. Bills for goods that have been sold at discounts from regular list prices should be collected. By such devices the children may be made familiar with the meaning of "discount" and with the fact that it is in very general use in practical business.

203. Reasons for giving discounts. — A keen interest may be developed in the question as to why business concerns should sell at a discount instead of asking in the first place the price which they expect to get and then selling for this price or not at all. It will soon develop that discounts are given for a variety of reasons. On the part of retail dealers the most common reason for giving discounts is that they wish to sell off goods that are getting out of season. Palm beach suits and straw hats are sold at a discount in August and fur coats in March. Christmas goods are sold at a discount in January, etc. Before taking inventory, which is usually done once a year, merchants like to dispose of as much of their goods as possible.

Wholesale dealers who sell goods to the retailer give discounts for a variety of reasons. Their standard price lists, or catalogues, are printed, say, once a year. During the year prices may fluctuate considerably from causes entirely unforeseen at the time the catalogue was made. For this reason the prices in wholesalers' catalogues are put sufficiently high to meet the emergency of possible high prices. If prices do not go up or if they go down, sales must be made at prices much below those given in the standard lists. Again, a wholesale dealer may wish to charge different prices to different customers even if the sales are made on the same day. Suppose one customer of doubtful financial standing wishes to buy a small bill of goods on the longest possible term of credit and that another customer wishes to buy a very large bill of goods for spot cash. For very good reasons the latter customer will get a lower price than the former. That is, he will get a larger discount.

The child should be made to understand that the expense of selling to the large customer is smaller in proportion to the amount of goods sold; that the amount of the bill sold on credit to the customer of doubtful standing may be lost altogether; that in any case there is likely to be some expense involved in collecting such bills; that in the mean time the money is tied up; and that, in many cases, it can not be counted among the seller's assets with a very high degree of confidence. All these are good reasons why the buyer for cash receives a larger discount than the buyer on questionable credit or even on very good credit.

Other elements may enter. The wholesaler may be selling his first bill of goods to a very desirable customer and may be willing to make a particularly low price to make the sale.

For reasons like these wholesale dealers are in the habit of quoting special confidential rates of discount to their customers. It is seldom indeed that goods are sold wholesale at the prices given in the price lists.

There are certain articles, such as schoolbooks, fountain pens, phonographs, automobiles, etc., which are advertised to be sold retail at certain prices. The wholesaler or manufacturer must then make reductions from these prices when he sells to the retailer, for otherwise the retailer would get nothing for his work and expense of handling the goods.

Surely such facts as those just mentioned should be known to well-educated American citizens, and there would seem to be no place in the curriculum of the elementary school as opportune for teaching them as in the class in arithmetic at the time when the subject of discount is taken up. Further, if these matters and others to be mentioned in sections that follow are understood fully, the problems on discount that occur in arithmetic or in practical life will present no difficulty. A considerable share of the total time given to the subject of discount should be given to the understanding of these fundamental elements.[1]

204. How discount is computed. — The amount of discount may be stated by giving the regular price and the reduced price, as "dining chairs, regular price $12.00, now reduced to $8.00." Or it may be stated by giving the regular price with a certain fraction off, as "men's overcoats, regular price $45.00, ⅓ off." It may be stated by giving the regular price with so many per cent off, as "women's dresses, regular price $90, 40% off"; or by simply giving the number of per cent off without specifying the regular price, as "33⅓% off on all small rugs."

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[1] "There may be days of teaching when not a figure is used during the arithmetic period. The social setting, the business institution, which calls for the calculation, is studied as carefully as the process of calculation." Suzzallo, *op. cit.* (p. 165), p. 19.

The most usual form gives the regular price and the number of per cent off. The per cent off is computed on the regular price as a base. Thus, if an article is offered at $60 with 25% off, the reduction is 25% of $60, or $15, making the reduced price $45. That is, in a problem on discount the regular price is the *base*, the rate discount is the *rate*, and the amount by which the price is reduced is the *percentage*. The selling price is of course the regular price less the discount. All these matters should be made perfectly familiar to the pupils.

205. Discount series. — On goods sold by the retailer to the final consumer it is not usual to give more than one discount, while on goods sold at wholesale two or more discounts are the rule rather than the exception. Suppose an article is listed at $100. A regular discount of 30% may be given because $70 is all that the market price warrants at the time of the sale. If instead of buying the article on the customary term of credit, say 90 days, the customer is ready to pay cash, he will get another discount which will be computed not as a certain rate per cent of the original price of $100, but as a certain rate on the reduced price of $70. The propriety of computing the second reduction as a rate per cent of $70 instead of $100 is obvious. If the customer had taken the 90 days credit, the deferred payment would have been $70 and not $100, and hence paying it at once would entitle him to a discount on the $70. If the second discount is at the rate of 2%, it will amount to 2% of $70, or $1.40.

Not infrequently several successive discounts are given on the same article. Thus there may be a regular discount of 30%, because the market price is low as compared to the list price, a second discount, say 10%, because goods are damaged, a third discount, 5%, because of a very large order, and a fourth discount, 2%, because of cash payment. These discounts are entered on the bill as follows: "Discounts, 30%, 10%, 5%, 2%." The method of deducting these discounts must be made clear to the child. This method is a matter of business usage, and the only way he can find what it is is by reading about it or hearing someone tell about it; he can not possibly reason it out for himself.

Suppose the list price is $100 and the discounts are 30%, 10%, 5%, 2%. After deducting the first discount there is $70 left. The second discount is 10% of $70, or $7. Hence $70 – $7, or $63, is the amount left after deducting the second discount. The third discount is 5% of $63, or $3.15; after the deduction of which there is left $59.85 ($63 – $3.15). The fourth discount is 2% of $59.85, or $1.20; after the deduction of which the net amount of the bill is $58.65.

206. The order of discounts immaterial. — It makes no difference in what order a series of discounts is taken. In the preceding section 30% was deducted, then 10% of what was left after the first deduction, then 5% of what was left after deducting the second discount, and finally 2% of what was left after deducting the third discount. The result would have been the same if, for instance, 10% had been deducted first, then 30%, then 2%, and then 5%. This rather surprising statement can be proved most easily by noting that the remainder, after say 30% is deducted, is 70% of the original bill. That is, the amount left after the first discount is deducted is $100 × .70. From this 10% may be deducted by taking 90% of it. Hence the amount left after the second discount is deducted is $100 × .70 × .90. In this manner it follows that the amount left after the 5% and the 2% have been deducted is $100 × .70 × .90 × .95 × .98. If the discounts are taken in the order 10%, 30%, 2%, 5%, the amount left would be $100 × .90 × .70 × .98 × .95, which is obviously the same as in the former case. It is now easily seen that exactly these same factors would be present in the final product no matter in what order the discounts are taken. The truth of this conclusion should be *verified* by actually computing two or more series of discounts in different orders.

207. A single discount equal to discount series. — The problem of finding a single discount equivalent to a given discount series which has often been regarded as difficult may be robbed of its terror by assuming a list price of $100, and proceeding as in the example in section 205. In that example it was shown that a discount series of 30%, 10%, 5%, 2% reduced a bill of $100 to $58.65, which is 58.65% of $100. Hence the single discount equivalent to this series is 100% – 58.65%, or 41.35%.

In this manner many such problems can be solved orally. Thus, it is seen at once that the series 25%, 10% is equivalent to a single discount of 32.5%; the series 40%, 5% is equivalent to a single discount of 43%; and that the series 20%, 10%, 5% is equivalent to a single discount of 31.6%. The practical reason for finding a single discount equivalent to a discount series is that in some cases a buyer has the choice of several different discount series and he naturally wants to select the one offering the greatest reduction. Thus one dealer may offer an article at 30%, 20%, and 10% off, while another dealer offers the same article at the same list price but with 40%, 15%, and 3% off.

208. Occupations in which discount is important. — Single discounts are of importance to people of all occupations, for there is scarcely a person who does not buy at a discount some time or other. The use of a discount series is confined almost wholly to those who deal in merchandise. However, all that they really *need* to know about a discount series is that according to general business custom each discount is a certain per cent of what is left after the preceding discounts have been deducted. If one has this information, knows elementary arithmetic, and has common sense, he will have no trouble with such problems involving discounts as are likely to occur in practice.

The relation,

discount = list price × rate discount,

will of course enable him to solve the three possible problems on single discount.

REVIEW QUESTIONS

1. What is meant by "commercial discount," or simply "discount"?

2. What are the reasons why merchants and wholesalers offer goods at a discount? How can children be made to understand these reasons? (§ 203.)

3. What is the base on which simple discount is computed? (§ 204.)

4. Describe a series of discounts. Why are several successive discounts given? How are these discounts computed?

5. How can it be shown that the order of a series of several discounts is immaterial? How can one discount be found which is equivalent to a given series of discounts? What are reasons for finding such single discounts? (§§ 206, 207.)

CHAPTER XIV

PROFIT AND LOSS

209. Definitions of profit and loss. — The difference between the cost of an article and the selling price is called profit if the selling price exceeds the cost, and loss if the cost exceeds the selling price. It is by no means clear, however, just what elements should be included under "cost." In the case of merchants, though there are many exceptions the following represents widespread usage: Cost includes the purchase price, freight, and cartage. That is, it represents the cost of the article laid down in the buyer's place of business. On the other hand, there are many items of expense connected with an article from the time it is bought until it is sold which are not included under cost. The merchant must carry insurance and pay rent, part of which falls on the article in question. Light, heat, a working force in the place of business, interest on many investments — all these are partly chargeable to the particular article. However, these items are not included under "cost" as the term is usually used by the merchant.

Under the influence of the practice of *cost accounting* the manufacturer has come to include under cost practically *all* the elements of expense connected with the production of an article, though here too usage differs. Frequently certain elements are charged not as "cost" but as "overhead." In farm accounting the "cost" is usually made to include practically all elements of expense. This is due to a widespread effort to find the "cost of production" of farm products.

It should be kept clearly in mind that the meaning of the term "cost" can not be determined by a course of reasoning or by discussion. We must learn in what sense the term is used in actual practice. A lively classroom discussion will serve to bring out this fact as well as the usual meaning of the terms "cost," "profit," and "loss."

210. Why there must be profits. — People engage in business for the purpose of making money. Consider the case of a man running a small store: He has invested his money in the business; he spends his own time in it, frequently working harder than those who work for wages or salary. If he were not in business for himself, he would be working for a salary, and he would put out his money at interest. Moreover, there are always chances that the business may turn out to be unprofitable in which case the business man may lose his own salary, the interest on his investment, and possibly part or all of the money invested. To set off against a certain income in case he should work for a salary and against the risk of loss, the business must hold out reasonable prospects of a considerable net gain over and above all expenses. This net gain must of course come out of the profits.

If the term "cost" is used to include only purchase price, freight, and cartage, the so-called "cost of doing business" must also come out of profits. This includes all expenses of the business, except those just mentioned. Rental of buildings, wages and salaries, insurance, advertising, loss from damaged goods and from uncollectible debts, and many other items must be paid from profits.

In the case of large concerns, such as great factories, railways, steamship lines, etc., profits are necessary to pay sufficient dividends to obtain new capital for extensions and improvements and to form a surplus to tide over periods of depression or to meet accidental disaster.

211. Profit computed on buying price, or cost, as a base. — The chief practical difficulty which the child encounters in solving problems on profit and loss arises from the lack of clear understanding as to what is the base on which the profit or the loss is computed. He may state the rule glibly enough and still fail miserably for this very reason. The more general usage is to compute the rate of profit (or loss) as a certain per cent of the cost. This may be brought home most effectively by means of numerous simple examples to be solved orally. (It will of course be made clear that the words "profit" and "gain" are used with the same meaning.)

"If an article is bought for $1.00 and sold for $1.50, what is the gain per cent, and why?"

Answer: "50 %, because 50 cents (the gain) is 50% of $1.00 (the cost)."

"What is the gain per cent if an article costing 40 cents is sold for 80 cents, and why?"

Answer: "100%, because 40 cents (the gain) is 100% of 40 cents (the cost)."

In such questions the "why" is important. The child should state, as in the answers above, which number represents the gain, which the cost, and, by implication, which number is used as the base.

The fundamental relation,

$$\textit{profit (or gain)} = \textit{cost} \times \textit{rate of profit},$$

will of course be developed and then used consistently. When these matters have been attended to the whole subject will be well in hand and the problems should be solved without further rules or explanations and, as in the case of other one-step problems, without analysis. See page 319.

212. Profit computed on the selling price as a base. — In some cases, especially in industries in which labor or skill contributes a large share of the value of the product, the selling price and not the cost is used as the base on which the profit is computed. Thus a druggist may charge $1.00 for filling a prescription while the materials used cost him only 10 cents. In such a case the druggist uses the selling price as the base and says that his profit is 90%. If he used the cost of the material as the base, his profit would be 900%.

In some parts of the country, especially in the newer sections, profit is computed on the selling price in nearly all lines of mercantile business; in such localities, however, a dealer in real estate who buys a building for $10,000 and sells it for $11,000 speaks of his rate of profit as 10%. That is, he uses the cost as the base.

Children should be encouraged to make inquiries for themselves to find out how business men in the community compute profits. A large variety of usage will be discovered. In some cases the dealer adds his "cost of doing business" to the other costs and computes the rate of profit on this sum. In this case "profit" is used to indicate "net profit" or "net gain."

213. Marking price, discounts, gains. — Problems combining cost, marking price, discount, and gain are excellent for fixing the meanings of these terms. We are all acquainted with the old problem:

> "Goods are marked 25% above cost and sold at a discount of 25%; what is the rate gain or loss?"

The difficulty lies solely in the vagueness with which the terms are apprehended. We solve the majority of our problems by leaning on the apparent form; the problem *looks* somewhat like one solved in the text. Indeed a great many pages in our very best textbooks consist of a problem solved as a model and of other problems which may be solved largely by imitating the form of the model. That is why miscellaneous problems are so much harder than the carefully sorted out groups that precede. How unpopular such problems are! It has often been said that the human being is lazier in his head than in any other part of his being. However that may be, it is certain there is a very widespread tendency to avoid careful examination of a statement with a view to ascertaining its exact meaning. It is this, evidently, which makes miscellaneous problems unpopular. Our views on those fundamental questions of education which were discussed in Chapter I will determine whether we do all that we can to get the children to dig out such problems for themselves or connive at their avoiding them.

214. Occupations in which "profit and loss" is of importance. — The principles and usages discussed in this chapter are of prime importance to all who are engaged in any kind of mercantile business or in manufacturing, and they are of rapidly increasing importance to the farmer. They are also of importance for general citizenship in a social organization in which public opinion greatly influences economic relations. Such matters should surely be touched upon at some point in the work of the grade schools, if for no other reason than to call the attention to the fact that "profit" in business does not exist solely for the purpose of satisfying the greed of the business man. At present there is no course on elementary economics in these schools (perhaps there never can be) and the course in arithmetic is probably the most convenient and appropriate place in which to discuss such matters.

REVIEW QUESTIONS

1. What is meant by "profit" and "loss"? What different meanings are attached to the word "cost" in computing profits? (§ 209.)

2. What items must be paid from profits? Explain why a business can not continue unless it shows a profit. (§ 210.)

3. What is used as the base when profit is computed as a certain rate per cent? (§§ 211, 212.)

4. Upon what does the choice of base in computing profit depend? How does it vary for different parts of the country and for different kinds of businesses? (§ 212.)

5. What is the difference between "gross profit" and "net profit"?

CHAPTER XV

COMMISSION

215. Definition of commission. — As regards mode of compensation, there are two distinct bases on which men are employed. In one case the worker receives a stated amount for his time more or less irrespective of what he accomplishes. Such compensation is called wages or salary. In the other case the worker is paid a certain percentage of the amount of business which he does. Thus a real estate agent sells a piece of property and receives as his pay a certain per cent of the selling price. All compensations which are fixed as a certain per cent or fraction of the amount involved in a transaction are called *commissions*.

The difference between the merchant who buys and sells goods and the agent who sells on commission is that the latter never becomes the owner of the property which he sells. If a man buys my used automobile and then sells it, he is doing the work of a regular merchant; if he finds a buyer for my car and receives as his pay 10% of the selling price, he is doing the work of an agent, and his pay is a *commission*.

216. Kinds of business done on commission. — Every day large volumes of farm products, such as butter, eggs, grains, and meat animals, are sent from the country districts to feed the inhabitants of the cities. A large share of these products is sent to "commission merchants" who sell to local dealers, to slaughter houses, to exporters, and others. The commission merchant receives a certain per cent of the selling price and sends the rest of it to his country customers. In every town and city there are "real estate agents" who sell land and buildings on commission; insurance agents receive as remuneration a certain per cent of one or more of the yearly premiums; a commission is paid for finding buyers for all sorts of commodities, such as used automobiles, musical instruments, etc.

The pupils should be encouraged to make a list of all kinds of work in their community which is done on a commission basis. In this way the subject may be made real to them. They will feel that they are studying a part of the world's serious work and not simply a vexation peculiar to the schoolroom.

217. Basis on which commission is computed. — When property of any kind is bought and sold on commission, the commission is a certain rate per cent of the amount involved in the transaction. There are some exceptions to this rule, the most important being the "brokerage" charged for buying and selling stocks and bonds and certain other standard commodities. Brokerage (see Chapter XVIII) is charged as a fixed amount for each unit bought or sold irrespective of the price obtained.

Some cases of commission sales may require explanation:

Suppose a commission agent receives a lot of eggs and sells them for $175. Suppose there are expenses for cartage, etc., amounting to $5, and that his rate of commission is 5%. Does he receive 5% of $175 (the selling price) or 5% of $170 (the selling price less expense)? The answer is that he receives 5% of $175.

Such discussion as is suggested by the above will lead to the fundamental statement about commission:

commission = amount of sale or purchase × rate of commission.

The purpose is to give real force and meaning to this general statement. When that has been done, problems involving commission will need no further explanation.

Paradoxical as it may seem, it is true that the best way to understand problems on commission as well as many other topics involving the application of arithmetic to business is not to solve such problems, but to get a thorough insight into some of the principles according to which the business is conducted. Furthermore the child is not likely to forget this information. If on the other hand he were to solve many problems according to specific rules without any intimate acquaintance with the situation which gives rise to them, he would be perfectly certain to forget all about the whole subject in a short time.

218. Occupations in which "commission" is important. — On reflection it will appear that problems involving commission are likely to present themselves in the course of a lifetime to a surprisingly large number of people. Besides those who are actually engaged in buying and selling on commission, there is the multitude of those who at one time or another will avail themselves of the services of agents working on commission. Farmers whose products are sold on commission, anyone who sells a piece of real estate through an agent — these and many others need to know the essential facts about commission. Even those who sell their effects at auction pay a commission to the auctioneer.

REVIEW QUESTIONS

1. Make a list of the kinds of businesses that are done on a commission basis in your community.

2. What is the basis on which commission is computed, (a) in case of a sale; (b) in case of a purchase?

CHAPTER XVI

INTEREST

219. Prevalence of loans and interest.— Many children have had some experience with interest by the time they begin the study of it in arithmetic. Some of them have had savings accounts at the bank, and they know that the bank pays a few cents each year for each dollar which it keeps. A boy of seven exclaimed with surprise that the bank paid him something for keeping his money when he thought he might have to pay the bank for taking care of it. They may know cases in which money was borrowed to build a house or buy a farm and that interest was paid on such loans. Some have been drawing interest on Liberty Bonds. Children should be encouraged to note and bring to the schoolroom all actual instances they can find of loans on which interest is being paid. A little prompting will reveal a vast body of loans. Probably the schoolhouse in which the children are studying was built with borrowed money. Roads, bridges, street pavements, public buildings, etc. are usually built with money that has been borrowed and on which interest is being paid.

Without using actual instances it should be pointed out that merchants borrow money from the banks with which to buy goods and pay the loans when the goods have been sold. Farmers borrow money to put in their crops and carry them through the summer and pay the loans when the crops are sold. In all such cases interest is of course paid. Bring out the fact that it is not the poor only who borrow; that in fact the well-to-do borrow more than the poor, and those who are in active business borrow more freely than those who are not.

Paying interest on borrowed money is a universal practice and no one can borrow in a business way without doing so. Any one who has money to lend will always receive interest.

The subject of interest will be considered at least twice in the grades, and the substance of sections 219–221 may be brought in partly the first time interest is studied and partly later.

220. Profit and interest. — Not only is interest paid whenever money is borrowed, but the element of interest enters into a great many situations where there is no actual borrowing or lending. Instances will illustrate: (*a*) A man having some money invests it in a business of his own. He expects the profits of this business to pay, besides all other expenses including a salary to himself, interest on the money invested. (*b*) A man with capital puts up a large building in the city. He does this because he believes the rental from the building will be sufficient to pay interest on the money invested, besides paying for all repairs, running expenses, and the depreciation of the building. (*c*) A man who buys a farm and equips it with stock and machinery will consider that he loses money unless the income from the farm is sufficient to pay interest on his total investment, besides covering all expenses of running the farm including pay for his own labor.

The important point is that none of these enterprises would be undertaken unless there were prospects of normal interest on the total investment. This applies to every industrial undertaking whatsoever — the building of railways, steamships, warehouses, office buildings, and to investments of all sorts. Hence it follows that in every business, interest must be figured on the amount of money invested whether the money is borrowed or not. This is a fact of such general importance that all should know something about it. Besides laying the basis for successful work in arithmetic, the teacher may feel quite confident that imparting information of this sort helps to produce intelligent citizenship.

221. Reasons for paying interest. — The reason that interest is paid can readily be understood even by young children. Suppose anyone having good security could borrow freely without paying interest. A case like this would then be possible: A man has money enough to buy a lot and build a house on it but not enough to grade the lot and plant it with trees and shrubs. After building the house, he gives the house and lot as security and borrows enough to beautify his grounds thereby very materially increasing the desirability of his home. In case the borrowed money is to be repaid, this man could, under our supposition, borrow from someone else, for his security is unquestioned. Under this arrangement the enhanced beauty of this home could be enjoyed forever without any trouble or outlay on the part of the owner.

Under such conditions a farmer could borrow money to drain wet places, plant fruit trees, and in various other ways make his farm more productive without any cost to himself except the trouble of borrowing the money. Children will take delight in finding instances in which people having property, which could be used as security, could borrow money to enrich their lives without any expense to themselves, if money could be borrowed without interest.

The obvious fact is, of course, that if interest were not paid the amount of money that could be borrowed would be very slight indeed. Someone has said that the man who "invented interest" was very smart. Interest was not "invented" any more than the law of gravitation; it was "discovered." It is an economic law with which it is as useless to quarrel as with the laws of physics and chemistry.

222. Base on which interest is computed. — The base on which interest is computed requires discussion in order to fix the idea. To illustrate the difficulty we need only to point out that pupils often have trouble with a problem like this: A $100 bond paying \$5.00 a year is bought for \$105.00. What is the rate of interest on this investment? In this case as in many others that have been noted, it is not sufficient to *say* that interest is computed on the amount loaned or invested, nor is it sufficient to get the pupils to *say* this. The real content of the statement must be made clear by illustrations and questions. The difficulty of reading into verbal statements their full and complete meaning is greater than many of us appreciate. This is well illustrated in our debates in the United States Senate on the ratification of treaties.

223. Element of time in computing interest. — If the rate of interest is 6%, this means that 6% of the *amount loaned* is paid for *one year's use* of the money. Frequently notes given for loans specify that the interest is — % *per annum*. If no further statement is made, it is understood that the total interest for one year is payable at the expiration of one year from the time the loan is made, or at the time the loan is due, if that is in less than a year. If the rate specifies that the interest is — % per annum *payable semiannually*, this means that half of a year's interest is payable in six months and the other half at the end of the year, and so on as long as the loan runs. If the interest is payable *quarterly*, one fourth of a year's interest is payable at the end of three months, another fourth at the end of six months, and so on.

The total interest on a loan depends upon the time it is to run. If the time is one year, then the interest is found by multiplying the principal (the amount loaned) by the rate. That is, in this case,

$$\textit{interest} = \textit{principal} \times \textit{rate}.$$

If the time is other than one year, the product, "principal × rate," must be multiplied by the time of the loan, the unit of time being one year. Thus, if the time of the loan is 6 months, the "time" is $\frac{1}{2}$; and if the time of the loan is 5 months, the "time" is $^5/_{12}$. In all cases,

interest = principal × rate × time.

The element of time in the applications of percentage is peculiar to problems involving interest. After the fundamental fact of interest itself has been made clear, this is the element upon which attention should be centered. "If the loan runs for one month, what is the time?" *Answer:* "$^1/_{12}$" (not one month). "If the loan runs for 7 months, what is the time?" *Answer:* "$^7/_{12}$". By means of such questions, stating the time in terms of months, years, or fractions of years (not in days), and by means of simple problems, the fundamental equation of interest given above may be given real meaning to the child. When this has been done, we pass to the case when the time is given in days.

224. The year of 360 days. — In computing interest when the time is given in months, it has been assumed tacitly up to this point that all months are equal. This assumption is now made explicit and is extended by saying that each month is regarded as 30 days and the year, therefore, as 360 days ($12 \times 30 = 360$). This method of counting time for the purpose of computing interest is in very common use. When the time is given in days or in months and days, there is no difficulty. If a loan runs for 54 days, the time is $^{54}/_{360}$ year $= {}^3/_{20}$ year. If the loan runs for 3 months and 9 days, the time is 99 days, or $^{11}/_{40}$ year.

The problem of finding the time between two given *dates* requires attention. Between days of the same date in any two consecutive months, the time is regarded as 30 days though in reality it may be 28 days, 29 days, 30 days, or 31 days. Thus between Feb. 15 and Mar. 15 there are 28 days in ordinary years, and 29 days in leap years. Between Apr. 15 and May 15 there are 30 days, and between May 15 and June 15 there are 31 days. From Jan. 1 to Jan. 20 is 19 days (not 20 days). That is, the day on which the loan is made and the day on which it is paid are not *both* counted. The time from Feb. 6 to April 28 is 2 months, 22 days, or 82 days, or $^{41}/_{180}$ year. If a loan is made on the 31st of a month, it is regarded as being made on the 30th of the month. Some curious results may follow from this usage. Thus, if a loan were made May 30 and paid June 1, the time would be one day though in reality it was twice that, and, if a loan were made Feb. 28 and paid Mar. 1, the time would be 3 days though in reality it was only one day.[1]

225. Direct method of computing interest. — Any problem in computing interest may be solved *directly* by using the relation,

$$\textit{interest} = \textit{principal} \times \textit{rate} \times \textit{time},$$

provided the "rate" is expressed as a common or a decimal fraction and the "time" as a fraction of a year. It is perhaps more convenient to express the rate as a common fraction. Thus

$$6\% = \frac{6}{100},\ 6\tfrac{1}{2}\% = \frac{6\frac{1}{2}}{100} = \frac{13}{200},\ \text{and } 5\tfrac{3}{4}\% = \frac{5\frac{3}{4}}{100} = \frac{23}{400}.$$

The "direct" method has the advantages that it is very easily learned and is equally convenient for all rates of interest. To know how to compute interest by this method, it is only necessary to understand the fundamental statement, "interest = principal × rate × time," and to know how to state the time in terms of years as described in section 224. This method has the disadvantage that it results in an undesirable form (see section 198). Those who make many computations of interest, unless they use interest tables, usually use one of the forms described in the following sections.

226. Bankers' method of computing interest. — The subject of interest is of sufficient importance to warrant the introduction of special methods. The following example is solved by the "bankers' method."

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[1] Loans for as short periods as one day are often made by banks to brokers who borrow on "call loans" and settle when they close accounts with customers.

EXAMPLE: Find the interest on $7800.00 at 6% for 44 days.

SOLUTION: 78.00 (interest for 60 days)
39.00 (interest for 30 days, ½ of 60 days)
13.00 (interest for 10 days, ⅓ of 30 days)
5.20 (interest for 4 days, 1/15 of 60 days)
57.20 (interest for 44 days)

EXPLANATION: Since 60 days is ⅙ of 360 days, it follows that 6% for one year is 1% for 60 days. Hence the interest at 6% for 60 days is obtained by moving the decimal point two places to the left. This amount ($78.00) is put down first as a general fact for reference. We then endeavor to find a combination of numbers whose sum is 44 such that each number can be found by a succession of simple operations on 60. In this case 30 is ½ of 60, 10 is ⅓ of 30 (or ⅙ of 60), and 4 is 1/15 of 60. Thus we see at once that the interest for 30 days is ½ of $78, or $39; the interest for 10 days is ⅓ of $39, or $13; and the interest for 4 days is 1/15 of $78, or $5.20. Adding these amounts, we get $57.20, the interest for 44 days.

When the rate is other than 6%, we first find the interest at 6% and then the interest at the required rate.

EXAMPLE: Find the interest on $585 at 6½% for 97 days.

FIRST SOLUTION

First find interest at 6%.
5.85 (interest for 60 days at 6%)
2.925 (interest for 30 days at 6%)
.585 (interest for 6 days at 6%)
.098 (interest for 1 day at 6%)
9.458 (interest for 97 days at 6%)
.788 (1/12 of 6% = ½%)
10.25 (interest at 6½%)

SECOND SOLUTION

5.85 (60)
2.925 (30)
.585 (6)
.098 (1)
9.458 (97 da. at 6%)
(1/12) .788
10.25 (6½%)

In practice the words in parentheses to the right are omitted, only the numbers of days being put down. A good form for this example is shown above at the right. The matter in parentheses is self-explanatory.

To find interest at 7%: find interest at 6% and then add $\frac{1}{6}$ of it. The following suggest methods of finding interest at other rates: 8 = 6 + $\frac{1}{3}$ of 6; 9 = 6 + $\frac{1}{2}$ of 6; 7½ = 6 + $\frac{1}{4}$ of 6; 5½ = 6 − $\frac{1}{12}$ of 6; 5 = 6 − $\frac{1}{6}$ of 6; 4½ = 6 − $\frac{1}{4}$ of 6; 4 = 6 − $\frac{1}{3}$ of 6; 3 = 6 − $\frac{1}{2}$ of 6. These include all rates that are likely to occur in practice. The following may also be used: 5¾ = 6 − $\frac{1}{24}$ of 6; 5¼ = 6 − $\frac{1}{8}$ of 6; 6¼ = 6 + $\frac{1}{24}$ of 6; 6¾ = 6 + $\frac{1}{8}$ of 6.

In practice it is best to carry out the interest in each step to three decimals. With very rare exceptions this will insure correctness to the nearest cent in the final result.

A variety of combinations may be used to obtain the desired number of days. Thus 44 days may be obtained by taking 30 + 15 − 1, or 60 − 15 − 1. To check the result, the same problem may be solved twice using different combinations of days.

Interest problems can be solved very rapidly by this method when one becomes accustomed to it. The writer has seen a young man solve 56 miscellaneous problems of this kind in 75 minutes, solving each one twice to check the results. The young man was a good university student and by no means an expert calculator. It is very doubtful whether such problems are solved more rapidly by the practical banker if we include the time required to pick up the interest table, turn to the proper page, and put the final result on paper. However, the use of the table probably requires less mental effort and will be preferred by those accustomed to it.

227. The six per cent method. — When using the six per cent method, the interest at six per cent is found first as in the case when using the method described in the preceding section. The steps are as follows: (*a*) find the interest on $1.00 at 6% for the given time, (*b*) multiply the interest on $1.00 by the principal to find the interest at 6% for the given time, (*c*) find the interest at the given rate.

The first step is carried out by using a table like the following:

0.01 = interest for 2 mo.
0.005 = interest for 1 mo.
0.001 = interest for 6 da.
0.002 = interest for 12 da.
. . . .
0.00⅙ = interest for 1 da.
0.00⅓ = interest for 2 da.
0.0005 = interest for 3 da.

By judicious extension and combination of these, the interest on $1.00 at 6% for any number of days can readily be obtained. Thus, the interest for 44 days is .005 + .002 + .00⅓ = .007⅓ (44 = 30 + 12 + 2). Then the interest on $7800 is 7800 × $.007⅓ = $57.20. (Compare page 350.)

228. Using the interest table. — To find the interest on $7800 at 6% for 44 days by means of an interest table, we should first turn to that part of the table which gives the interest at 6% and then in this part to 44 days. We should find that the interest on $7000 is $51.33 and the interest on $800 is $5.87. Adding these, we get the required interest. The tables used in the banks contain the interest at all usual rates, such as 3%, 3½%, 4%, 4½%, etc., but not such rates as 5¾% or 6.8%, which occur at times in practice.

$$\begin{array}{r} \$51.33 \\ \underline{5.87} \\ \$57.20 \end{array}$$

229. Comparison of methods for computing interest — There would seem to be little doubt that the *bankers' method* (see section 226) is shorter and less liable to error than the *six per cent method* (see section 227.) The process of multiplication required in using the latter method must be performed on separate paper if the work handed in is to appear at all businesslike. One way in which the comparative brevity of two methods may be tested is to count the number of figures required by each method in solving the same problem. A comparison of the two methods for finding the interest on $7800 at 6% for 44 days is shown on page 353.

The bankers' method requires 15 figures and the six per cent method at least 30 figures. (The figures in parentheses are not counted as they are not essential parts of the solution.)

The writer has done considerable computing by both of these methods, and it is his belief that the work may be done at least twice as rapidly by the bankers' method as by the six per cent method.

BANKERS' METHOD

78.00	
54.60	(42 = 7 × 6)
2.60	(2 = 1/30 of 60)
57.20	(44)

SIX PER CENT METHOD

0.007	(interest for 42 days)
0.00⅓	(interest for 2 days)
0.007⅓	(interest for 44 days)
7800	$7800 × .007⅓ = $57.20
7⅓	
54600	
2600	
57.200	(interest on $7800)

It remains to compare the bankers' method and the use of the interest tables. If the rate is 6%, there is no doubt that the work may be carried out more rapidly by using the bankers' method. The number of figures required by the two methods is 15 and 11 respectively (using the above example) but the time required to find the numbers in the table will be a great deal more than the time required for the extra 4 figures. This example is believed to be a fair illustration of what happens in solving any problems where the rate is 6%. One advantage of the tables is that the interest at any rate given in the table may be found just as rapidly as if the rate were 6%. This is not true either of the bankers' method or the six per cent method. The writer believes, however, that a good computer can find interest at any rate about as rapidly by the bankers' method as by using an interest table.

It is of course obvious that even if interest could be found much more rapidly by using a table, it would still be necessary to learn some other method. Interest tables are available for the use of only a very few persons and very many people need to compute interest at some time in their lives. The writer believes it would be well if the schools were to standardize on the bankers' method of computing interest.

When the subject of interest is first introduced, the *direct method* (see section 225) should no doubt be used to make clear the fundamental fact that interest is proportional to the time of the loan. Obviously the direct method is most easily learned. Perhaps the best arrangement would be to use the direct method in the sixth grade, where interest is usually studied very slightly, and to use the bankers' method in the seventh grade. The usual objection to learning two methods would seem of little force here. The direct method is an obvious application of the fundamental principle of interest and requires no explanation when that principle is understood. Moreover solving some examples by this method helps to fasten this principle in the pupil's mind. The bankers' method should be regarded, as in fact it is, as a special method for carrying out the work indicated in the fundamental relation, "interest = principal × rate × time." The use of special devices for carrying out an operation is usually not objected to on the ground that it requires the learning of two ways for carrying out the same work. All short cuts are of this nature. We first learn to multiply by 125 in the usual way and later to obtain the same result by moving the decimal point three places to the right and dividing by 8. No one would object to a short cut for dividing by 33⅓ on the ground that another method has already been learned. The bankers' method may be regarded as a short cut in computing interest.

230. Finding the rate, principal, or time. — In problems in which the rate of interest or the principal is required, the time is nearly always one year. The problem is thus identical with the finding of the rate or the base in ordinary percentage when the other two elements are given. Problems in which the rate is required are of very frequent occurrence and will be considered in detail in the next chapter. (See section 235.)

Problems in which the principal is required are of more frequent occurrence in practice than is usually supposed. The following is a genuine example: "A man paying $720 a year rent for a house finds that taxes, repairs, and depreciation amount to

$270 a year. How much can he afford to pay for this house if the rate of interest is 6½%?"

Finding the time, when the principal, rate, and interest are given, is required very rarely, and no special attention should be given to this problem. If such a problem should occur, the fundamental relation, "interest = principal × rate × time," will always furnish a method of solution. A problem of this type is involved in finding the date on which the balance of an account should be paid.

231. Local applications of interest. — In practically every community in the United States interesting local applications of the subject of interest may be found. Very likely the schoolhouse in which the children are working was built with borrowed money. Find out how much the building cost and the rate of interest actually paid on the bonds issued. It will be interesting to find how much the interest amounts to each year for each child attending the school. Find the cost of the county court house, and using the prevailing rate find the interest which the county pays on this building each year. A college builds a great auditorium to be used on state occasions, such as commencement, concerts, etc. Suppose the building cost $300,000 and that it is used 20 times in one year. At 4½% interest on the cost of the building how much does this amount to for each time the building is used? Problems of this kind taken from the immediate neighborhood will not only furnish good drill but will help to inform the pupils on certain very important matters in regard to which there is a widespread and dense ignorance. Some day these pupils will have to vote on the issue of bonds for various kinds of public projects, and it would seem to be the duty of the schools to point the way to intelligent understanding of such matters. The purely financial considerations, which are matters of arithmetic, are important elements in such cases.

REVIEW QUESTIONS

1. Make a list of different kinds of loans in your community on which interest is paid. (Public loans for erecting schoolhouses, etc.; loans on real estate; ordinary bank loans to business men, etc.) (§§ 219, 231.)

2. How does the element of interest partly determine the rate of profit which a business must pay? (§ 220.)

3. Give examples showing that, if money could be borrowed on good security without interest, many would want to borrow and few would want to lend. How would the amount saved by people be affected if money could not be loaned at interest? (§ 221.)

4. What is the base on which interest is computed? How does the element of time enter in computing interest? What is the general equation for computing interest? In this equation what unit of time is used? (§§ 222, 223.)

5. Describe the direct method of computing interest. How many days are counted as one year? 45 days is what fraction of a year? (§§ 224, 225.)

6. Explain the bankers' method of computing interest. How is interest for 48 days found by this method? How is interest at 7% found? (§ 226.)

7. Explain the six-per cent method of computing interest. (§ 227.)

8. Compare the various methods of computing interest, (a) as to ease of learning; (b) as to rapidity with which work can be done; (c) as to appearance of the solutions.

9. How may the general equation of interest be used to find any one of the elements, principal, rate, time, and interest, when the other three are given? Give practical examples of each of these problems. (§ 230.)

CHAPTER XVII

SAVINGS AND INVESTMENTS

232. What is meant by savings. — A boy has a regular weekly allowance and earns some money besides. One month his total income is $7.50, of which he spends $2.50 for things he needs and puts $5.00 in the savings bank. It is clear that he has *saved* the $5.00. During one year the members of a family use $1,000 less than their total income for living expenses and loan this amount to a neighbor who is building a house. Again it is clear that this family has saved $1,000.

There are many cases, however, in which it is not so easy to decide whether the disposal made of a part of one's income should be regarded as a saving or as current expenditure. Suppose the boy, instead of putting the $5.00 in the savings bank, had bought a good pair of skates. He might argue that the skates will last many years, that he has paid for them, and is now the owner of additional property worth $5.00 and that he has really saved that much. It would be difficult to prove that the boy's argument is not valid. Suppose the family, instead of loaning the $1,000, had used the money to buy a piano and an oriental rug. At the end of the year they might argue that they had really saved the $1,000, pointing to the rug and the piano as evidences of their saving. Again it would be difficult to prove the argument wrong. Suppose further that during this year the furniture used by this family had depreciated $500 due to natural wear. Should this amount be deducted from the supposed saving of $1,000? Cases of this sort show that it is not so easy as it appears at first sight to decide whether a person is actually saving or not.

To decide definitely whether a family has made a saving during the past year, it would be necessary to have a list showing the fair money value of all their property (an inventory) at the beginning of the year and another such list at the end of the year. These lists together with statements of their debts (if any) at the beginning and at the end of the year would enable them to decide whether or not there has been an actual saving. If the net value of their property at the end of the year is greater than at the beginning of the year, there has been a saving and otherwise not.

Time may not permit the actual making of such inventories but discussion will bring out the essential points and may serve to start the pupils thinking about matters which are of great importance in practical life and which are too frequently neglected even by adults. Interesting cases come to hand readily: A man who has $2,000 in the bank borrows $4,000 additional to build a house and uses $1,000 of his year's income to fix up the grounds and put the house in shape for occupancy. Did he effect a saving or not? The children themselves will think of many such questions.

233. Investments. — In every grade-school course in arithmetic there are many problems on income from investments. At some point there should certainly be a discussion about what is meant by "investment." The children should be guided to use the word in the usual business sense. In the first place "savings" and "investment" are by no means identical in meaning. A family buying a rug and a piano for one thousand dollars may be regarded as "saving" this amount but they would not be regarded as "investing" it. The traditional miser who collects money in the form of gold coins and counts them daily is not an investor. "Investing" consists in so placing one's savings that they may reasonably be expected to earn an income either in the form of money or some other valuable property. Buying property which simply contributes to the buyer's comfort or pleasure and which he would otherwise forego is not regarded as an investment. It is of course impossible to draw a *definite* line between "investment" and other expenditures and this is of no particular consequence. Examples will serve to make clear the essential meaning. Loaning money at interest; buying bonds or stocks; putting money into a store, a shop, or a factory; building houses for rent; buying a farm or farm equipment — all these are regarded as investments. Buying or building a house in which to live would be an investment since the rent saved would presumably earn interest on the cost.

The aim of a classroom discussion of this type is not to fix a carefully formulated definition of "investment" but to indicate in a general way what is meant by this term and to point out the very large variety of ways in which money can be invested.

234. What kind of investments to avoid. — Many so-called investments are pure gambles. Buying stocks in oil companies just starting and having only fair prospects of finding oil or in mining companies on whose properties there is no definite evidence of a good yield of paying ore is simply taking a "long chance." On a comparatively small number of such investments very large incomes have been realized, but the majority of them have resulted in little or no profit and in many cases in the loss of the capital invested. The cost of selling such stocks is usually very great and those who invest in them are usually people who have little or no real information about the value of the property they are buying. The glowing prospects held out by the glib-tongued salesman, who is paid a substantial commission on stocks that he sells, snare the innocent and confiding by the thousands. Teachers and ministers are said to be particularly numerous among the victims.

A very good general rule for those who are not experts in financial matters is that *any investment which offers unusually high returns is probably bad.* Any business concern that is forced to borrow at higher rates of interest than the ordinary is regarded as unsound by those who understand such matters, for otherwise the money could be borrowed at normal rates. Stocks of concerns that are known to be earning enough to pay dividends are taken by banks and large investment companies who sell them to their customers through regular channels.

Many investment proposals are pure frauds. Shares are sold in mines which are solid granite mountains; in orange groves which are only pleasant dreams; in irrigated tracts in the West which are as dry as the Sahara. There are hundreds of men in the United States prisons for using the mails to advertise worthless stocks.

235. Rates of income on investments. — From the purely arithmetic point of view, by far the most important problem is to find the *rate of income* when the amount invested and the net income are given. The only type of investments in connection with which this problem does not occur is the straight loan which is made at a fixed rate of interest. A hundred-dollar bond which yields a fixed income of six dollars per year (6% on the par value, see Chapter XVIII) is very seldom bought for exactly one hundred dollars, and hence the rate of income on such investment would usually not be exactly 6%. Suppose such a bond were bought for $104.50. Then the rate of income would be 5.74% because $6.00 is 5.74% of $104.50. Even this is not absolutely correct: Suppose the bond is due in ten years. The investor will then realize $100 for the bond and not the $104.50 which he invested. Hence, the real rate of income is somewhat less than 5.74%. Suppose that at the same time a hundred dollar bond which yields a fixed income of five dollars is bought for $90.75. The apparent rate of income would be 5.51%, since $5.00 is 5.51% of $90.75. However, when this bond is due the investor receives $100 for it instead of the $90.75 which he invested. The real interest realized on this investment is therefore somewhat higher than 5.51%. To find the real rate of income on such investments is far too difficult for a course in arithmetic. A solution of this problem is given in our colleges and universities in courses on the Mathematical Theory of Investments. In practice the real rate is found by the use of special tables which are at hand in all the larger banks and investment houses.

While finding the real rate is beyond our field, the character of the problem should be pointed out. It is easy to understand that the rate found by the methods of elementary arithmetic is always too high for bonds bought above par and always too low for bonds bought below par. It is also easy to understand that the difference between the rate thus found and the real rate is less when the bond has a long time to run and also less when it is bought very nearly at par. When the bond is bought at par, the method of arithmetic gives the real rate. The rate found by the elementary methods is nearly always a more or less close approximation.

The finding of such approximate solutions is however of real value. Knowing the facts just stated, we can make a very close estimate of the real rate and an effective comparison of rates (which is the important matter).

While the difficulty just noted enters into a great many problems on investment, there are other types of problems in which the investment and the income for one year or less are given. In such cases the problem consists simply in finding the rate of interest when the principal, interest, and time are given (see section 230).

236. Safe rates of interest. — It has been pointed out that investments which offer exceptionally high returns nearly always involve exceptional risks (see section 234). What, then, is a safe rate? The answer depends on many elements. Tax exempt securities, such as government bonds or municipal bonds, may yield a considerably lower rate than bonds of industrial concerns which are regarded as perfectly safe. The tax exemption feature is important to large investors who pay a high rate of income tax on all income from securities that are not exempt. For the small investor this is of little importance. A man with a very large income may find it to his advantage to buy a government bond yielding 4.5% rather than the bond of a railroad yielding 6%. The small investor would buy the 6% bond.

Rates of interest differ quite widely in different parts of the country. They are in general lower in old and well-settled communities than in the newer parts of the country. Farm mortgages, which are regarded as among the very safest of investments, may yield five to six per cent in New York or Illinois and seven to eight per cent and even higher in Montana or Idaho.

The only safe rule is to avoid unusually high rates. They are more dangerous to the lender than to the borrower. Not more than a generation ago a "woman's bank" which offered 2% a month was started in Boston. Anyone with the most elementary information about investments would give such an institution a wide berth.

Those who invest money in their own business usually expect a higher rate of return from their investment than those who loan money. The business man himself carries a much higher risk than those who loan him money. If the business fails, all debts must be paid first, and he gets what is left. Hence no new business should ever be undertaken unless there are reasonable prospects for a higher rate of profit from the investment than the normal rate of interest on good securities. Hence it is that those who succeed in business make high profits from their investments. The average is brought down to the proper level by the losses of those who fail. No exact data seem available on this point, but it is probable that the optimism with which men enter on new ventures is responsible for a very low average. All the railways in this country taken together have paid less than savings banks' rates on all the capital invested in them since they were built.

237. Place in curriculum where savings and investments should be studied. — If any topic in arithmetic should be treated by the "spiral" method, it is the topic of savings and investments. Problems on savings should be given in every grade from the lowest in which formal arithmetic is studied. Questions relating to the income from investments should be introduced as soon as simple interest has been mastered and possibly earlier. The discussions which are suggested in this chapter should not all be introduced at any one time but from time to time through several grades. It is unnecessary to say to the competent teacher that only such topics should be considered as can be made comprehensible to the child. It is surprising, however, how much of this matter can be placed within the child's reach and made really interesting provided the teacher herself is properly equipped with information, personal enthusiasm, and a belief, grounded in intelligent understanding, in the importance of this subject.

Discussion may be very much worth while even if the child can not understand fully all that is considered. We all understand but in part and see but in part. Ideas may be started which will be the beginnings of maturer study and reflection in later years. The boy or girl who is aware of the existence of many practical problems not yet fully comprehended, but whose importance is felt to be great, may be a rare product of our schools, but nevertheless a very desirable product.

238. Importance of this subject in the curriculum. — The writer feels justified in urging further the importance in our schools of the subject of savings and investments. If it be admitted that a person's attitude on such questions as the value of accumulating some property *versus* living from hand to mouth can be influenced in an important degree by early education, then the value of this subject in the schools is measured by the value, to society and to the individual, of saving and investing.

Civilized society of the modern type is possible only where large amounts of property are saved. Every item of property except land and streams in their natural form represents a saving made voluntarily by individuals. This is unqualifiedly true of all property created by private enterprise. It is very largely true of improvements made by the State or the municipality, for such improvements are usually made with money obtained through the sale of bonds and bonds can be sold only to those who have saved. There are not a few who openly defend the thesis that saving should not be encouraged. They say, "let old-age pensions and sick benefits take care of those who by age or illness are incapacitated for work." These well-meaning enthusiasts misunderstand the situation hopelessly. If everybody were to do as they advise there would not be a house in which to live, a factory in which to work, a railway or even a highway on which to travel, a machine or an animal to transport our goods — and there would be no goods to transport. If these people are rational at all in their position, they must mean that so far as *they* are concerned they will not save; they propose to use up from day to day, or at least within a short period, the total results of their work. If saving is necessary, they propose that others shall do it. Such a position is at least understandable though its moral implications may not excite our admiration. "Help us this day to bear our share of life's burdens" is a prayer with which these people are not acquainted.

In our civilization, organized as it is on the basis of individual freedom and undivided responsibility for one's self and one's family, there is nothing of greater importance in a person's economic life than saving part of one's income; for it is only by so doing that one can become the owner of a part of that vast body of products of human efforts which has been brought into existence not wholly for the purpose of satisfying present wants. If somehow we could get our boys and young men, and our girls and young women too, to develop a definite purpose of saving systematically we should be doing them the greatest personal service. The young man who invests part of his income regularly with the idea that he may never expend this money but that every hundred dollars saved will yield him five or six dollars additional each year throughout his life will be doing one of the most important things for his own larger and more permanent satisfaction and happiness that he can possibly do.

For one thing the person who saves and invests every year throughout his life say 10% of his total income each year will in the course of a reasonably long life actually spend more money than he who uses up his total income each year, for in not many years the interest on the accumulated savings will exceed the 10% which he saves each year. Tables showing this in detail have been constructed and are highly instructive. But from the point of view of personal welfare this is by no means the most important consideration. The person that has saved feels a security which otherwise he could not have. He has placed a cushion between himself and adversity whereby its rude shocks may be absorbed or whereby they may be averted altogether. He can not be put out on the bare ground; he is saved from the distressing worry that if misfortune should overtake him it would reduce him and those depending on him to misery and possibly to an abject condition that is revolting to all people of fine spirits.

Youth is optimistic and is inclined to give little thought to such matters. Perhaps this is true of our teachers also, the great majority of whom are well on the sunny side of middle life. Certain it is that in few respects have our schools failed more seriously in "preparing for life" than in this one respect of inculcating the purpose of saving and thus achieving a measure of economic independence.

Let us remember throughout that mere "preaching" has little effect on conduct. Imparting information that is definite and reliable, the directing of the attention to inevitable alternatives, the pointing out that "streaks of luck," or "windfalls" come seldom if ever — through these we may hope to bring young people to the right attitude on this important question.

239. Examples on saving. — Few people realize how rapidly systematic saving and investing result in the accumulation of considerable property. The following examples which represent approximately actual conditions will be a revelation to many and are very likely to stimulate systematic saving. To save space only every fifth year is given in the solution of the first four examples.

EXAMPLE 1. A young man in his twentieth year begins working at his trade and earns on an average $1200 a year until he is 65 years old. Each year he saves and invests one tenth of his total income at 6%. Find for each year the income from interest, the total income, the amount saved, the amount expended, and the total savings by the end of the year.

ANSWERS:

AGE	TOTAL INCOME	YEARLY EXPENDITURE	YEARLY SAVING	TOTAL SAVING	INTEREST
20	$1200.00	$1080.00	$120.00	$ 120.00	
25	1229.18	1106.26	122.92	609.29	$ 29.18
30	1266.52	1139.87	126.65	1235.03	66.52
35	1304.96	1179.46	130.50	1879.76	104.96
40	1344.57	1210.11	134.46	2544.09	144.57
45	1385.40	1246.86	138.54	3228.57	185.40
50	1427.47	1284.72	142.75	3933.87	227.47
55	1470.81	1331.67	147.96	4808.51	279.63
60	1515.46	1372.10	152.46	5561.75	324.56
65	1561.47	1413.76	157.09	6357.88	370.85

EXAMPLE 2. The same as the above except that the saving is one fourth the total income:

ANSWERS:

AGE	TOTAL INCOME	YEARLY EXPENDITURE	YEARLY SAVING	TOTAL SAVING	INTEREST
20	$1200.00	$ 900.00	$300.00	$ 300.00	
25	1292.74	969.55	323.19	1868.87	$ 92.74
30	1392.65	1044.49	348.16	3558.97	192.65
35	1500.28	1125.21	375.07	5379.72	300.28
40	1616.23	1212.17	404.06	7341.18	316.23
45	1741.14	1306.85	434.29	9453.22	541.14
50	1875.61	1406.71	468.90	11729.48	675.61
55	2020.59	1515.44	505.15	14181.63	820.59
60	2176.75	1632.56	544.19	16823.33	976.75
65	2344.98	1758.73	586.25	19669.20	1144.98

EXAMPLE 3. At the age of nineteen a boy enters a business establishment at a salary of $1200 a year. He receives a yearly increase of $100 until he reaches $4000, at which salary he continues until he retires at seventy years of age. He saves and invests one tenth of each year's income at 7%. Find for each year the income from interest, the total income, the amount saved, the amount expended, and the total saving by the end of the year.

ANSWERS:

AGE	TOTAL INCOME	YEARLY EXPENDITURE	YEARLY SAVING	TOTAL SAVING	INTEREST
20	$1200.00	$1080.00	$120.00	$ 120.00	
25	1749.64	1574.68	174.96	884.13	$ 49.64
30	2318.08	2086.27	231.81	1918.62	118.08
35	2907.19	2616.47	290.72	3250.55	207.19
40	3517.42	3165.68	351.74	4886.37	317.42
45	4149.32	3734.38	414.93	6833.77	449.32
50	4602.94	4142.65	460.29	9073.75	602.94
55	4766.24	4289.62	476.62	11422.97	766.24
60	4935.42	4441.88	493.54	13856.70	935.42
65	5110.60	4599.54	511.06	16376.73	1110.60
70	5284.89	4756.40	528.49	18884.09	1284.89

Example 4. The same as the preceding except that one third of the total income is saved.

Answers:

Age	Total Income	Yearly Expenditure	Yearly Saving	Total Saving	Interest
20	$ 1200.00	$ 800.00	$ 400.00	$ 400.00	
25	1870.53	1247.02	623.51	3059.60	$ 170.53
30	2621.83	1747.89	873.94	6900.15	421.83
35	3466.21	2314.14	1152.07	12097.94	766.21
40	4420.88	2947.25	1473.63	18914.75	1220.88
45	5485.16	3657.82	1828.34	27330.81	1785.16
50	6477.15	4218.10	2159.05	37546.96	2477.15
55	7268.92	4845.95	2422.97	49121.81	3268.92
60	8157.22	5438.15	2719.07	62107.91	4157.22
65	9136.31	6090.97	3045.44	76668.74	5136.31
70	10280.06	7853.37	3426.69	93141.34	6280.06

The results in Example 4 are represented graphically on page 415. It will be instructive to compare the amounts expended yearly by those who save with the amounts which would be expended if they did not save. We find that for about twenty years these people have less to spend than they would have if they did not save; but after that there is a larger amount available for expenditure though the saving continues in increasing amounts. In each case the total expended is considerably larger than it would have been if each year's income had been completely expended.

Example 5. A professional man at the age of thirty-four has $8000 invested at 5%. His yearly earnings up to the age of sixty-six are shown in the column to the left. He invests at 5% approximately one half the amount by which his total yearly income exceeds $2500. Thus, if the total income is $5000, he invests one half of $2500 ($5000 – $2500), or $1250. The actual investments are shown in the fifth column. Find for each year the income from interest, the total income, the amount expended, the amount saved, and the total saving.

Answers:

Age	Earned Income	Total Income	Expenditures	Yearly Savings	Total Savings	Interest
35	3800.00	4200.00	3400.00	800.00	8800.00	400.00
36	4150.00	4590.00	3590.00	1000.00	9800.00	440.00
37	4250.00	4740.00	3640.00	1100.00	10900.00	490.00
38	4400.00	4945.00	3745.00	1200.00	12100.00	545.00
39	5000.00	5605.00	4005.00	1600.00	13700.00	605.00
40	5250.00	5935.00	4235.00	1700.00	14400.00	685.00
41	6000.00	6720.00	4620.00	2100.00	16500.00	720.00
42	6200.00	7025.00	4725.00	2300.00	18800.00	825.00
43	6250.00	7140.00	4840.00	2300.00	21100.00	940.00
44	6500.00	7555.00	5055.00	2500.00	23600.00	1055.00
45	7000.00	8180.00	5380.00	2800.00	26400.00	1180.00
46	7400.00	8720.00	5620.00	3100.00	29500.00	1320.00
47	8200.00	9675.00	6175.00	3500.00	33000.00	1475.00
48	10000.00	11650.00	7150.00	4500.00	37500.00	1650.00
49	9560.00	11375.00	6975.00	4400.00	41900.00	1875.00
50	11600.00	13695.00	8195.00	5500.00	46400.00	2095.00
51	10800.00	13120.00	7820.00	5300.00	51700.00	2320.00
52	14600.00	17185.00	9685.00	7500.00	59200.00	2585.00
53	15300.00	18260.00	10260.00	8000.00	67200.00	2960.00
54	12600.00	15960.00	8960.00	7000.00	74200.00	3360.00
55	10400.00	13310.00	8310.00	5000.00	79200.00	3710.00
56	13600.00	17560.00	10060.00	7500.00	86700.00	3960.00
57	18200.00	22535.00	12535.00	10000.00	96700.00	4335.00
58	15800.00	20635.00	11635.00	9000.00	105700.00	4835.00
59	13200.00	18485.00	10485.00	8000.00	113700.00	5285.00
60	11000.00	16585.00	9585.00	7000.00	120700.00	5585.00
61	12800.00	18835.00	10835.00	8000.00	128700.00	6035.00
62	10500.00	16935.00	9935.00	7000.00	135700.00	6435.00
63	8400.00	15185.00	8685.00	6500.00	142200.00	6785.00
64	9300.00	16410.00	9410.00	7000.00	149200.00	7110.00
65	6500.00	13960.00	8486.00	5500.00	154700.00	7460.00
66	6900.00	14635.00	8635.00	6000.00	160700.00	7735.00

In Example 5 the total expended is less than it would have been if the total earned had been expended each year. However, this man is in a position to retire and use over $8000 yearly for the rest of his life without using any part of his accumulated property.

In discussing these problems it should be pointed out that the savings furnish a good living in old age, in most cases without using any of the principal that has been saved.

Problems like these may easily be made by the teacher or pupils, and it will be found that they are of absorbing interest. There is no table by which these problems may be solved, and hence teacher and pupil may rest assured that they are practical from every point of view. It is true that they are laborious, but that is true of very many "real" problems.

REVIEW QUESTIONS

1. Discuss the meanings of the words "saving" and "investment." Are savings always invested? (§§ 232, 233.)

2. Discuss "safe" and "risky" investments; also discuss "safe" rates of interest. (§§ 234, 236.)

3. How may the rate of income on an investment be found? (§ 235.)

4. Discuss the place in the curriculum where the subject of saving should be considered. (§ 237.)

5. Discuss the importance of studying saving and investing as a part of the school curriculum. (§ 238.)

6. Give concrete (preferably local) examples of the effects of saving and investing. These may be modeled after the examples in section 239.

7. Discuss the effect upon a county or a state or the whole nation if no one were to save. Would there be any new buildings or railways or machinery, etc.?

CHAPTER XVIII

STOCKS AND BONDS

240. Importance of stocks and bonds in the curriculum. — None of the applications of percentage thus far considered has been seriously recommended for complete exclusion from the grade curriculum, but not a few who are prominent in educational matters have advocated that stocks and bonds be entirely omitted. On analysis, the arguments for such omission would seem to break down entirely. It has been said that the subject is of practical importance only to a small number, that it is very difficult, and that it is not of natural interest to children. I do not believe any of these statements are true.

Stocks and bonds are beyond comparison the most important single form of investment. During the War the Government of the United States sold over twenty billion dollars of bonds; railways represent very nearly another twenty billions of stocks and bonds; the States, counties, cities, and school districts in this country have issued very many billions of bonds; the total of bank stocks runs far into the billions; and other private concerns have issued scores of billions of stocks and bonds. The capital stock of the U. S. Steel Corporation alone is one billion dollars. It is estimated that from one sixth to one fourth of the total property of the United States is represented by stocks and bonds which are owned by several millions of people. A number of corporations have over one hundred thousand stockholders. If it be granted that it is important to have some knowledge of the ways in which savings may be invested, it follows that information about stocks and bonds is of very general value. Indeed such information is necessary for the intelligent discharge of one's duty as a citizen, even if he never buys a bond or a share of stock. Proposals for bond issues are constantly being referred to the voters, and some information about bonds is necessary for intelligent action.

It is entirely erroneous to suppose that the subject of stocks and bonds is necessarily difficult. This will appear more clearly in the sections that follow. From a fairly extensive experience in conducting courses for teachers, I feel safe in hazarding the assertion that the practical difficulty in teaching stocks and bonds lies not in the inherent difficulty of the subject but in a remarkably widespread lack of information on the part of the teachers. Frequently I have said to classes of teachers: "If you will give me your aggressive attention for fifteen minutes, we shall learn to understand the essential elements of the subject of stocks and bonds." At the expiration of this period the majority of such a class will have a clear understanding of all that is required to solve such problems as should be attempted in the grades. A few will fail because they are slow in their mental operations and some because they have made up their minds that the subject is too hard and hence make no effort.

The statement that the subject possesses no inherent interest for children is no more true of stocks and bonds than of many other subjects whose total omission has never been urged. With proper presentation of the information that is needed about stocks and bonds themselves and with a clear explanation of the terms used, the subject may be made just as interesting as the other applications of percentage.

241. Organization of a stock company. — The subject of stocks and bonds may be introduced by a study of the formation of a stock company, some local concern being studied as a concrete example. The concern has "capital stock," say of $500,000. This stock is divided into shares (usually) of $100 each. If a man owns one thousand such shares, this means in effect that he owns a one-fifth interest in the company and is entitled to one fifth of the profits paid to the stockholders. Suppose that in one year this company earns $40,000 over and above all expenses. It may be decided to keep $10,000 in the company's treasury as a "surplus" and to divide the $30,000 among the stockholders. In that case the man owning one fifth of the stock gets one fifth of $30,000, or $6,000, as his *dividend*. This dividend is his income for the year from the money which he invested when he bought the stock. To make a more lasting impression, a stock company may be organized in the class, stock issued, and stock certificates written out and distributed among the members. A blank stock certificate may be obtained from a local bank.

242. Common and preferred stock. Cumulative preferred stock. — An example will make clear the difference between preferred and common stock. A certain company has issued and sold a total of $200,000 preferred stock and $300,000 in common stock. That is, there are two thousand shares of preferred stock and three thousand shares of common stock. The rate of "dividend" on the preferred stock is fixed at six per cent. This means that the owner of one share of the preferred stock will get six dollars each year *provided* the net earnings of the company are sufficient to justify paying this amount in dividends on all the preferred stock. No dividend can be paid on the common stock until the fixed dividend on the preferred stock has been paid in full. The owners of the preferred stock can not receive more than six dollars per share, no matter how prosperous the company becomes.

After the six dollars per share has been paid on the preferred stock, the remainder of the net earnings is divided among the owners of the common stock in proportion to their number of shares. If the earnings of the company are small, the owners of the common stock may receive nothing whatever while the owners of the preferred stock receive their six dollars per share. On the other hand, if the earnings are large, the owners of the common stock may receive much more than six dollars per share while the owners of the preferred stock are definitely limited to the amount specified in their stock certificates. We will consider the following cases:

(*a*) Suppose the net earnings of the company for one year are $4,000. This is not sufficient to pay dividends even on the preferred stock, and the whole amount may be put into the "surplus." The owners of the stock whether preferred or common get nothing whatever.

(*b*) Suppose the net earnings for one year are $15,000. Then six dollars per share ($12,000 in all) may be paid to the owners of the preferred stock; $3,000 will probably be put into the surplus, and the owners of the common stock will get nothing.

(*c*) Suppose the net earnings are $35,000. The owners of both preferred and common stock may receive six dollars per share ($30,000 in all), and $5,000 may be put into the surplus.

(*d*) Suppose the net earnings for one year are $100,000. The owners of the preferred stock will receive six dollars per share ($12,000 in all), and the owners of the common stock possibly twenty-five dollars per share ($75,000 in all). This would leave $13,000 to be put into the surplus.

Under case (*a*) where no dividend on the preferred stock is paid for a certain year the owners of the stock have no further claim for payments for this year. There is, however, another kind of stock called "cumulative preferred" which provides that if no dividend is paid for one year this must be paid as soon as the earnings make this possible. Suppose that in the above case the preferred stock were cumulative and that the earnings for the year following the year of no dividend were $30,000. Then the owners of the cumulative preferred stock would receive twelve dollars per share before anything could be paid to the owners of the common stock. In some cases unpaid dividends on such stock may accumulate over a period of years, but the whole amount thus accumulated must be paid before the holders of the common stock receive anything whatsoever.

243. Industrial bonds. — A bond is to all intents and purposes a promissory note. It specifies that a certain sum will be paid on a specified date to the holder of the bond and that up to that time a certain rate of interest will be paid on it. There is an essential difference between a bond and a share of stock. The man who owns a bond of an industrial organization has the same kind of a claim against it as any other creditor who has loaned it money.[1] A stockholder has a right to a share in the profits, if any, while the bondholder must be paid whether there are profits or not. A company which fails to pay interest on its bonds or which fails to pay the bonds when due thereby declares itself bankrupt, whereas a concern may fail for years to pay dividends to its stockholders and still go on doing business.

244. Municipal and government bonds. — There should be no difficulty in any school in the United States in finding local examples of municipal or State bonds. There are few counties or cities which have not issued bonds. State bonds are well known and every child has heard something about United States Government bonds. Such a bond is simply a promise by the Government or the municipality to pay a certain amount on a certain date and to pay a stated rate of interest up to that time. It may be advisable to discuss the provisions of the law in some states whereby the municipality is compelled to levy a certain tax on all property situated in it for the purpose of paying interest on its bonds and also to pay the bonds when they are due. It may also be worth while to say something about sinking funds, provided this is done in a simple, informal, and nontechnical manner.

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[1] It is of course understood that in special cases certain property may be mortgaged as a special security for the payment of bonds as well as for any other debt. Bonds advertised as "mortgage bonds" are secured in this way.

245. Debenture bonds. — The term "debenture bonds" occurs frequently in financial statements in the newspapers and in statements issued by industrial and financial organizations. A debenture bond provides for the payment of interest and of the bond itself out of a certain specified source of income which by the terms of the bond is pledged for this purpose. Thus a real estate company may issue bonds providing that the income from a certain building shall be used to pay the interest and that when the bonds fall due this building may be sold to pay the bonds. In such cases a mortgage on the building in question is executed and recorded. The company may fail entirely and all its general creditors may lose every cent the company owes them but the holders of the debenture bonds are safe provided the property specially pledged is sufficient to cover these bonds and the interest due on them.

246. Registered and coupon bonds. — It will be interesting to show the difference between "registered bonds" and "coupon bonds." If possible, specimens of these two kinds of bonds should be brought into the schoolroom. The name of the owner of a registered bond is "registered" in the books of the company, municipality, or government which issued it. Interest when due is sent to the owner in the form of a check. If a registered bond is sold, the name of the owner must be changed on the books of the concern which issued the bond. If such a bond is lost, a duplicate may be issued and an unauthorized possessor can collect neither interest nor the value of the bond itself.

A coupon bond on the other hand has attached to it small slips of paper, called *coupons*, which represent the value of the interest as it falls due from time to time. These coupons are cut off as they become due and are presented for payment. In practice they are deposited in a bank which sends them in for collection. A coupon bond may be sold without formality. Whoever has a coupon or a coupon bond in his possession may present it for payment when it is due.

Large bonds are usually in the form of registered bonds. Those who buy bonds with the purpose of selling them soon may prefer coupon bonds while those who buy bonds as a more permanent investment are more likely to prefer registered bonds. Coupon bonds must be kept in a safe place, such as a bank vault, where thieves can not easily get them or fire destroy them.

247. Par value and market value. — A bond always specifies that on a stated future date a certain sum will be paid to the owner of the bond. This sum is called its *par value*. In most cases a stock certificate specifies that each share of stock has a certain "par value," usually one hundred dollars.[1] If such a bond or share of stock is sold for more than one hundred dollars, it is said to be above par and if sold for less than one hundred dollars it is below par.

Stocks and bonds are seldom sold at par, the actual price paid depending on a large number of elements. In the case of a bond the rate of interest, the certainty that the interest and the bond will be paid promptly when due, the time which it has to run, the exemption or nonexemption from taxes on income derived from it, all have an influence upon the price at which it is bought and sold. The market value of a share of stock depends upon the rate of dividend paid on it and on the probability that the payments of dividends will continue. The stock of a company which is not making money will have a low market value, and the stock of a prosperous company will have a high value. Some railway stocks which in 1916 were sold close to par were worth less than twenty dollars a share in 1921.

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[1] In the last few years corporations have frequently issued stocks of "no par value". The value of such stock, like that of any other kind, is determined by the earnings of the corporation and the consequent amount divided among the stockholders. One purpose of "no par value" stock is to avoid paying taxes in certain cases where these are levied on the par value of the stock irrespective of the earnings or losses of the company.

Lists of the prices at which important bonds and stocks are bought and sold can be found in any large newspaper. The pupils will be interested to see the figures at which the stocks of a familiar railway, for instance, are actually being sold.

248. Dividends on stocks. — Pains should be taken to make clear just what is meant by "dividend" and "rate of dividend." If a stock pays a six per cent dividend, this means that *six per cent of the par value* of the stock is paid each year by the company to the owner of the stock. It does not matter in the least what the market value of the stock is. Failure to understand this thoroughly is the cause of a large share of the difficulty which pupils and teachers alike experience in solving problems on stocks and bonds. It should be explained too, as suggested above, that the dividend on stocks, and especially on common stocks, varies with the prosperity of the company — that they are high when earnings are high and low when earnings are low.

249. Interest on bonds. — It should be explained clearly that the interest on a bond is definitely specified in the bond, that under no circumstances will the owner of the bond receive more than this amount and he will not receive less, unless the issuer of the bond becomes bankrupt. For this reason the value of bonds does not fall as low in times of industrial depression or go as high in times of prosperity as does the value of stocks.

250. Bonds and stocks contrasted as to security. — Government and municipal bonds in the United States are practically absolutely safe. In our whole history only a few such bonds have ever been repudiated, and these were without exception issued under peculiar circumstances, many of which savored of fraud. Municipal bonds are now as safe as any investment that can be made. Back of these bonds stands the total property of the community and laws frequently provide that the issuance of such bonds automatically levies a special tax on the community which can not be removed until the bonds are paid.

Bonds issued by private concerns are always a *safer* investment than their stocks. The owner of stock may not only fail to receive dividends for a period of years, but his stock may in the end become worthless. The bondholder will lose only in case the total property of the corporation should be insufficient to pay its debts. It should not be inferred, however, that all stocks offer a low degree of security. There are railway companies which have paid the usual rates of dividends every year upward of seventy years; and there are banks which have paid dividends for even a longer period. All well-established corporations make great efforts to maintain their credit on a high level by making regular payments of dividends. Stockholders in a national bank, in case of failure of the bank, may not only lose the money they invested in the stock but an additional amount equal to the par value of the stock.

251. Brokerage. — Stocks and bonds are bought and sold on the "stock exchanges" throughout the country. Men engaged in trading on these exchanges are called *brokers*. A broker at times buys and sells for his own account; but his chief business is buying and selling for others, for which service he charges a commission, called *brokerage*. Brokerage is always computed as a certain rate on the *par value* of securities bought or sold and does not depend in the slightest degree on their market value. Simple as this is, it has caused no end of confusion for both pupils and teachers.

The standard rate of brokerage for buying or selling stocks and bonds is one eighth of one per cent of the par value, or 12½ cents for a share or bond of $100 par value. Hence the brokerage for buying or selling 100 such shares is $12.50 and for 1000 shares it is $125. This rate of brokerage is standard throughout the country except in some cases where small lots are bought or sold.

252. Rate of income from investments in stocks and bonds. — The problem of by far the most general importance in regard to stocks and bonds is finding the rate of income on an investment. It is exceedingly simple, all statements to the contrary notwithstanding. How much does a share of stock or a bond cost you and how much does it yield as a yearly income? If these questions are answered, the rate is easily found (see section 230).

EXAMPLE: A share of stock yielding $6 per year income was bought at a net cost of $110; what was the rate of income on that investment?

SOLUTION: $6 is 5.44% of $110. Hence the required rate is 5.44%.

The endless confusion which has reigned over this realm comes from the fact that we have not had in mind certain simple facts upon which the solution of the problems depends. We have been too indolent or too dull to make clear to ourselves that the rate of dividend on a stock or the rate of interest on a bond is computed as a certain rate per cent of its *par value*.[1] A "6% stock" pays $6 dividend *per share* of $100 par value, and an "8% stock" pays $8 dividend *per share*.

EXAMPLE: Find the rate of income on an investment in an 8% stock bought at a net cost of $125.62½ per share.

SOLUTION: $8 is 6.37% of $125.62½. Hence the required rate is 6.37%.

This problem, which is typical of those that arise under stocks and bonds, is as simple as any problem in finding the rate in the general subject of percentage, *provided* we know what is meant by an "8% stock." The real source of the confusion is, no doubt, that the word, "rate," is used with two different meanings in the same problem. The "rate of dividend" is a rate per cent of the par value while the "rate of income on the investment" is a rate per cent of the amount paid for the security. This, however, will cause but little difficulty when the attention is definitely directed to it.

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[1] In the case of *no par value stock* the rate of dividend is given as a certain amount per share.

The cost of a share of stock or a bond must include brokerage for buying. Thus, if the market value is $125.50 per share, the real cost to the investor is $125.50 + $.12½ = $125.62½.

253. The essential simplicity of stocks and bonds. — Too much stress can not at present be laid on the essential simplicity of the subject of stocks and bonds. Indeed the importance of discussing this simplicity is perhaps not so great in the case of pupils as in the case of teachers and principals. Many of the latter have a superstitious dread of the subject which the less sophisticated pupils have not yet acquired. When one mentions stocks and bonds in a conclave of teachers, a sigh goes round. If one has occasion to question them on the subject, many show very promptly that they do not know its essential elements.[1] The reason for this lack of information is, no doubt, that the affairs of their lives have not caused them to take interest in this subject and that when they studied it in school, instead of having the main facts pertaining to it made clear in some such manner as is urged in this chapter, they were given "rules" by which to solve the problems. The rules were never intelligible, were soon forgotten, and the future teacher was left an entire stranger to the whole subject. It is no wonder that their imaginations have infested this unknown realm with terrors among which young children should not be allowed to venture.

There is one proposition upon which we teachers seem to be agreed; namely, that any topic which causes the teacher any difficulty whatever must be far too difficult for the pupils. But precisely those things seem easy which we already know and those things which we do not know seem difficult. No doubt a Chinaman would regard our system of writing more difficult than his own. Hence it follows that it is very difficult to get the rank and file of teachers to present optimistically and enthusiastically to elementary classes any subject which they did not learn in early youth.

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[1] The writer has conducted classes on the teaching of arithmetic every year for the last fourteen years. The majority of the students in these classes have been grade teachers or principals of many years experience.

254. Problems to be included under stocks and bonds. — It is easy to find problems on stocks and bonds which will cause any amount of difficulty. A stockbroker's account with one of his customers is a case in point. All such matters which are of interest only to the broker and the speculator should be definitely excluded. As in the case of other topics, we should know just what we want to teach and then confine ourselves to that. Under the general subject of stocks and bonds we may not be far astray if we set out to impart such general information as is given in this chapter and, besides this, to solve the following three types of problems:

1. Find the cost of a certain number of shares when the market price is given (include brokerage).
2. Find the net receipts from the sale of a certain number of shares when the market price is given (deduct brokerage).
3. Find the rate of income on an investment in stock (or bonds) when the rate of dividend (or interest) and the market price are given.

REVIEW QUESTIONS

1. Discuss the importance of stocks and bonds as means for investing one's savings. To what extent is the ownership of wealth in the United States represented by stocks and bonds? (§ 240.)

2. How is a stock company organized? What are dividends and from what source do they come? (§ 241.)

3. What is meant by "common stock," "preferred stock," and "cumulative preferred stock"? In what order of priority are dividends paid on these kinds of stocks? (§ 242.)

4. What are industrial bonds? municipal or government bonds? debenture bonds? registered bonds? coupon bonds? (§§ 243–246.)

5. What is meant by "par value," "above par," "below par," "market value"? (§ 247.)

6. What is meant by "6% dividend on stock"? At this rate how much does a man receive in dividends if he holds one hundred shares of $100 par value? Does the amount he receives depend upon the price at which he bought the stock? (§ 248.)

7. Does the dividend on preferred stock depend upon the prosperity of the company? How? Does the dividend on common stock so depend? How? To what extent does the interest paid on bonds depend upon the prosperity of the company issuing them? (§§ 241, 242, 243.)

8. Compare, as to security, (a) municipal bonds; (b) industrial bonds; (c) industrial stocks. (§ 250.)

9. What is brokerage? How does brokerage differ from other commissions? (§ 251.)

10. What is the method of computing income on an investment in stocks or bonds? Is this rate usually the same as the rate of dividend or interest? Explain fully. (§ 252.)

11. What types of problems should be included under "stocks and bonds" in the course in arithmetic? (§ 254.)

CHAPTER XIX

BANKS

255. Savings banks. — It may be best to begin the study of banks by considering the savings bank. Many children have savings accounts and hence possess considerable information about the business usages that govern the conduct of a savings bank. These usages should be studied with a view to making such information definite and accurate within the limits set by immaturity of the pupils and the time available. Copies of the regulations in use by one or more local banks may be obtained. The meaning of "interest period" and "interest dates" should be made clear. If interest is paid (compounded) semiannually, the interest period is six months and runs from January first to June thirtieth and from July first to December thirty-first. In this case July first and January first are the interest dates. On each interest date the banks pay interest on the smallest amount on deposit during the preceding interest period. If this interest is not withdrawn by the depositor, it is added to his account and draws interest in precisely the same manner as the rest of the account. If interest is compounded quarterly, each interest period is three months, the first period of the year being from January first to March thirty-first. The interest dates in this case are the first days of January, April, July, and October.

256. Rates of interest paid by savings banks. — The rate of interest paid by savings banks will be found to be very low, varying in the main from three to four per cent. Comparatively few savings banks pay more than four per cent, and the greater part of the money on deposit in such banks draws less than this rate.

The question may be raised as to why the savings bank should pay such low rates of interest. The answer to this question may be approached by raising another question; namely, "What is the source of the bankers' profit from which he pays the interest?" It will be clear that the banker can not receive the money and leave it in his vaults. In order to pay interest on the deposits, pay the cost of running the bank, and make a profit for himself, he must loan the money at a higher rate of interest than he pays. Moreover, the banker is obliged by law to invest savings deposits in particularly high-grade securities which because of their very high degree of safety pay a low rate of interest. The funds on which a savings bank pays three and a half or four per cent interest may be invested by the banker at five or six per cent.

Finally the banker can not invest all the money on deposit in the savings bank. Money is continually withdrawn as well as deposited, and there must be a considerable cash reserve not only to pay normal withdrawals from day to day but to meet possible emergencies of unusual withdrawals. For these reasons the rate of interest on savings accounts must always be low. The child should be taught to look with suspicion on a savings bank which offers an unusually high rate of interest.

257. Practical importance of the savings bank. — The savings bank makes it possible to invest and get interest on very small amounts. A child may deposit a dollar a week or a dollar a month, or even less, and interest will be paid according to the rules stated in section 255. Investments of amounts less than $100 can not easily be made in stocks and bonds or any other standard security. Thus the savings bank provides a means for getting some return on small savings until they amount to enough to invest in a standard security yielding a larger rate of interest. In some cases an investor may place a considerable sum in a savings bank while looking for a favorable investment.

The savings bank acts as a stimulus for saving which has made systematic savers and investors of hundreds of thousands who otherwise would save nothing whatever. One measure of the importance of the savings bank is that of late years the total deposits in them have run above $6,500,000,000, which is about $60 for every man, woman, and child in the United States.

258. Commercial banks. — The commercial bank receives money on deposit and pays it back on demand without any notice whatever. Usually no interest is paid though large accounts (deposits) sometimes draw as high as two per cent interest. Commercial banks make loans, usually for short periods, to people of good financial standing, or on adequate security.

259. The checking account. — A detailed study should be made of the checking account. Deposit slips should be secured and made out, and the manner of depositing cash and checks in the bank should be explained fully. Children should be taught to make out a check properly and to endorse a check. The figures and the words stating the amount of a check should be written in such manner that the check can not easily be raised.[1] Thus in a check for $15.00 the numerals 15 should be written close to the $ sign thus, $\$15\frac{00}{100}$ and not thus, $\$\ 15\frac{00}{100}$, for in the latter case the check might easily be changed to read $\$215\frac{00}{100}$. The words indicating the amount of the check should be started very closely to the left end of the check to prevent the insertion of words such as "Two hundred" at the left end of the line.

It should be explained that checks will be cashed at the bank only when the maker of the check has sufficient money on deposit to cover the amount of it, and that money can not be withdrawn from a savings account by means of checks.

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[1] This is not as general an accomplishment as one might suppose. The writer has seen checks made even by teachers which might easily be raised. Endorsing a check on the wrong end is very common.

When a check is deposited in a bank, the person in whose favor it is made must "endorse" it. That is, he must write his name on the back of the check. This endorsement should be written on the *left end* of the check. The convenience of the use of checks should be pointed out. Suppose on the first of the month a man has twenty different bills to pay. The checks are made out, attached to the bills, enclosed in envelopes, and put into the mail, all in less than a half hour. There is no danger of loss for no one can cash a check until it has been endorsed by the person or firm to whom it is made out. Since a check is presented to the bank for payment and finally returned to the maker, it is a receipt for the money paid. Hence no other receipt is necessary.

Compare this method with the old method of paying in cash: The man paying his bills would start out with several hundred dollars and make personal visits to twenty places situated in different parts of the city. The custom of paying bills with checks is now very general especially in the United States.

260. Loans made by banks. — Practically the only source of income of a bank is the interest on loans which it makes. By far the greater part of the loans made by commercial banks are short-time loans to people who are in business. An example will illustrate: School textbooks are usually printed and bound in the spring and early summer and sold largely in the fall. The publishing house which pays for the books when received from the printer can not look for income from sales before October or November. Suppose that such a company needs $200,000 from July first to November first to carry its stock of books. If the company has good financial standing, the bank with which it does business will loan this amount taking the company's note for four months. The money is placed to the credit of the company on the bank's books and is withdrawn by means of checks as it is needed. The whole amount may be drawn out in a few weeks after the loan is made. Money received from sales is deposited in the bank and when the note is due it is paid by drawing a check against these deposits. Such borrowing would be done by the publisher no matter how rich he might be. Suppose this publisher had $200,000 extra which he might use for the purpose just stated. Instead of putting this amount into the book business he would invest it at say six per cent getting $12,000 a year interest. The interest charged by the bank for four months would be only $4,000. Hence a net gain of $8,000 a year would be realized by borrowing from the bank.

This example illustrates a very common business practice. Nearly all who are in business borrow from the banks, and the richest and best established companies are the greatest borrowers. Manufacturers borrow to buy raw materials and defray the cost of manufacturing and pay when the finished product is sold; merchants borrow when they buy goods and pay when the goods are sold; farmers often borrow to put in their crops and harvest them and pay when the products are sold. Such loans, which are called *commercial loans*, are made for definite business enterprises, and it is understood that they are to be paid from the proceeds of these enterprises.

It will thus be seen that the bulk of the loans made by commercial banks are quite different from those made by savings banks and by general investors who make long-time loans. The long-time investor furnishes funds for construction of buildings, railways, ships, roads, pavements; for farm betterment, and all kinds of permanent improvements; and also for capital that is needed permanently in carrying on business. The commercial banks furnish funds which are needed for comparatively short periods of time. It is true that commercial banks make some investments in long-time securities, but this is merely incidental to their main function which is to make short-time commercial loans.

261. Bank discount. — The commercial bank usually collects interest in advance. To revert to the example of the publisher borrowing $200,000 from July first to November first: When the loan is made, the bank deducts the $4,000 interest and credits the borrower with the difference between $200,000 and $4,000, which is $196,000. When the note is due the borrower pays the full $200,000. This method of collecting interest is called "bank discount," and the bank is said to "discount" the note. It should be pointed out that such notes are made so as to carry no interest. The essential difference between ordinary interest and bank discount should also be noted. If $200,000 is borrowed at six per cent ordinary interest for four months the borrower has the use of $200,000 for that time at an interest cost of $4,000, while if a $200,000 note is discounted at the bank the borrower has the use of only $196,000 at the same interest cost. In some sections of the country, especially the newer and less highly developed parts, banks collect interest at the end of the period the same as in ordinary loans. The rates of interest in such cases are nearly always higher. The vast majority of loans made by commercial banks in the United States are made, however, on the basis of bank discount. The method of computing bank discount is exactly the same as that of computing ordinary interest.

262. Method of studying banks. — In some schools play banks are organized, and the functions of the bank actually are carried out by the pupils. It is doubtful whether this pays in an elementary class. The child should be given accurate information about the *general* functions of a bank, while minute details should be avoided. It is perfectly possible, perhaps it is probable, that a class carrying on a fairly complete banking business may fail entirely to learn the important facts about a bank such as have been described in this chapter and which should be known by all well-informed citizens. Very many who have worked for years as clerks in banks have failed to obtain such information as may be imparted easily in the elementary schools without special organization and without much loss of time.

REVIEW QUESTIONS

1. What are the functions of a savings bank? What is meant by "interest period" and "interest dates"? On what dates do "interest dates" usually fall? (§ 255.)

2. What are the usual rates of interest paid by savings banks? Why are these rates comparatively low? On what amount does the savings bank pay interest? (§§ 255, 256.)

3. What are the functions of a commercial bank? What kind of loans are usually made by such banks? (§ 258.)

4. Describe a checking account. What are some of the precautions that should be taken in making out a check? (§ 259.)

5. Why do very rich business firms borrow large sums from the banks? (§ 260.)

6. What is meant by "bank discount"? In what respect does it differ from ordinary interest? (§ 261.)

CHAPTER XX

Paying Bills at a Distance

263. Why there are bills to be paid at a distance. — In nearly all courses of study some time is allotted to the study of postal money orders, express money orders, and even bank drafts, all of which relate to the paying of bills at a distance. In this connection some information should be given about the reasons for paying bills at a distance. This will be highly interesting to the children, will require very little time, and will serve to connect the study of arithmetic with other parts of the school curriculum and especially with geography.

The things in daily use in any family will be found to come from all parts of the world. Indeed a small number of these articles will be found to have been produced completely within the state in which the child lives. There are many states in which a yard of cloth has never been woven, a man's hat produced, or a suit of ready-made clothing or a piece of furniture manufactured. Farm machinery, hardware of all kinds, pianos, automobiles, cotton, sugar, tobacco, iron, copper, and other metals are produced in only a few states but are in daily use in every community. All these things must therefore be bought and paid for in distant parts. Such payments may be made by sending cash, personal checks, postal or express money orders, or bankers' checks. Payment may also be ordered by telegraph. We shall now consider these in turn.

264. Why distant payments are seldom made in cash. — The sending of actual cash is very expensive if properly guarded against loss and is seldom used in making distant payments. Small bills or even silver coins may be enclosed in an ordinary letter, but these should be registered to guard against loss. Amounts of five dollars and upward are seldom sent in this way, and the majority of smaller remittances are made in some other way.

265. Postal orders and express orders. — Payments of small bills, such as subscriptions for magazines and newspapers, and payment for books and other articles ordered by mail are often made by postal money orders or express orders. The charges made for such orders are given in any textbook on arithmetic or may be learned at the post office or express office. The rates for postal orders and express orders are the same.

Large amounts are never sent by money orders, because the cost is high, thirty cents per hundred dollars, and because it would be very inconvenient to send a large number of orders to pay one sum. A money order is never issued for more than $100 and express orders are never issued for more than $50. To pay $10,000 by money orders would require one hundred separate orders and would cost thirty dollars.

266. The personal check. — The most convenient method for paying a bill at a distance is to send a personal check, and this is being done with increasing frequency. The personal check has, however, certain disadvantages. If an article, such as a book, is ordered by mail from a distant part of the country and a personal check is enclosed in payment, those receiving it have no means of knowing whether the check is good and can not learn except by sending it through their bank for payment. In the case of great distances it may take as long as ten days before the bank will know whether or not the check has been paid. In some cases banks will not credit the depositor's account with checks on distant parts of the country before word has been received that they have been paid. The banks receive such checks "for collection."

In some cases a small charge is made for collecting "out of town" checks.

267. The bankers' draft. — Large bills are usually paid by "bankers' drafts." To illustrate: A merchant in Seattle who wishes to pay a bill in Chicago buys a draft on Chicago from his bank. The banker in Seattle has a regular checking account in some Chicago bank and simply makes out a check on this bank for the desired amount and sells it to his customer, who endorses the check to his Chicago creditor and sends it through the mails.

Banks all over the country have accounts with banks in the chief financial centers. If a bill is to be paid anywhere in the territory tributary to Chicago, a draft on Chicago will be used, and, if in territory tributary to New York, a New York draft will be used. Moreover, a bank draft on any of the great financial centers will usually be accepted by any bank in the United States.

268. Service of the banks in paying bills at a distance. — The use of bankers' drafts saves the trouble and expense of shipping immense amounts of cash from one section of the country to another. A simple case will help children to understand how this is really done. Suppose a merchant in Chicago owes one million dollars for shoes in Boston and that a Boston dealer in grain owes a million dollars in Chicago for a consignment of wheat. The Boston merchant buys a draft for a million dollars on Chicago at the same time that the Chicago merchant buys a draft for one million dollars on Boston. The banks have arrangements by which these drafts are set off one against the other so that both debts are paid without the transfer of a single cent in cash.

This is an example of what is constantly taking place on a very large scale. Throughout the year the North Central States are shipping great quantities of breadstuffs and meats to the New England and Middle Atlantic States, and these in turn are shipping clothing, shoes, and other manufactured goods to the North

Central States. Drafts are constantly being sent in both directions to pay for each individual shipment, and these drafts are set off against each other as far as possible. Cash is shipped only in case, for a considerable length of time, the total of the drafts sent in one direction is considerably greater than the total of the drafts sent in the opposite direction. Even this statement is not entirely true. The true statement is that cash is shipped into a region only when that region sends out a greater volume of valuable articles than it imports. Among these valuable articles must be included securities, such as bonds, stocks, mortgages, and payments of debts or for services which may be rendered to its citizens who are traveling or living in other parts of the world. The whole situation is too complicated for complete discussion, but examples such as are given above will serve to make clear the essential facts.

REVIEW QUESTIONS

1. Discuss the reasons why there are many bills to be paid at a distance. Why, for instance, do business men in Chicago have bills to pay in New York or Boston? (§ 263.)

2. Why are bills at a distance seldom paid in cash? Discuss fully how payments can be made without transfer of cash? (§ 264.)

3. What are the immediate means usually employed in paying such bills? Discuss the means whereby mutual intersectional indebtednesses are set off against each other. (§§ 265–268.)

CHAPTER XXI

TAXES

269. What taxes are spent for. — Numerous expenditures are made by what are called public corporations, such as towns, townships, counties, and cities, as well as by the governments of the various states and of the United States. Public schools are maintained by the town, city, or State; roads are built and kept in repair; court houses and city halls are erected for the safe-keeping of documents and the transaction of public business; courts of law are maintained; prisons are built and cared for. There are asylums for the insane, homes for the aged and destitute, and sanatoriums for the sick.

In the cities streets are paved and kept clean. If a heavy snow falls, men and teams or trucks are set to work to remove it. Garbage and ashes are removed; public parks are provided and kept in order; there is a police force to give protection against the lawless; and a fire department to guard against fires.

The army and navy are maintained. There are pensions for soldiers and their widows. Salaries are paid to State and United States officers from the President down. Rivers and harbors are improved, canals are dug, and many other public works are undertaken. Nearly one billion dollars is paid yearly as interest on the large debt of the United States Government. The money to defray these expenses is nearly all derived from taxation either direct or indirect.

270. Why taxes are increasing. — As civilization advances, there is an increasing number of things for which the town, city, State, or Federal Government must spend money. In primitive days we built our own roads, fought our own fires, and protected ourselves as best we could against thieves and murderers. All this is now done by the State. Extensive investigations are undertaken and expensive services are maintained with a view to protection against disease. The business of farming is studied in our agricultural colleges and by the Federal Farm Bureau at the expense of scores of millions each year. Experiments are carried out at public expense in many fields for the benefit of industry and trade. Better and more costly school buildings are erected. More and better educational facilities are constantly being provided for an increasing number of the young generation. It will be highly instructive to have children make a list of the purposes for which taxes are expended in their neighborhood.

271. Average per capita tax. — Before the war the average per capita expenditure of the United States Government was between \$7 and \$8. In the year 1921 this expenditure was about \$45 per capita, and the budget for 1923 is over \$30 per capita. It will be easy to find the per capita for the current year by dividing the amount of the budget for this year by the estimated population.

Per capita expenditures of State governments vary widely, some being as low as \$3 or \$4 and others running nearly as high as \$20. The total expenditure of local governments (county, town, or city) in some cases runs above \$50 per capita. The total yearly public expenditure in the United States is not far below \$100 for every man, woman, and child, and we shall not be so very far from the truth if we estimate that one sixth of the total income in the United States is taken as taxes in one form or another.

The total yearly expenditure by the State, the city, or county can easily be found and hence the per capita expenditures of each of these. The imparting of such information will consume little time in the schoolroom. The children will be glad to look up the data if they are told where to find them. The value for citizenship of such work is unquestionably great, and its value in correlating arithmetic with the practical affairs of life is equally great.

272. Sources of revenue. — Every person who owns property pays a tax on it with the exception of certain small amounts that may be exempt. Such tax is called a *property tax*. All persons having an income above a certain amount pay an *income tax* to the United States Government and in some states to the State Government. Corporations pay a *special tax* on their profits. Many business concerns pay a *license fee* for the privilege of doing business. A tax, called an *excise tax*, is levied on the manufacture and sale of such articles as tobacco, cigars, cigarettes, liquors, oleomargarine (if artificially colored), etc. There is a tax, called *import duty*, on a large number of articles imported into the United States. It will be very interesting to study the financial statement of the county or city in which the school is located. Such a statement will contain a complete list of all sources from which revenue is obtained.

273. The assessor. — In every county there is an official, called the *assessor*, whose duty it is to prepare a list of all property owned by each individual, partnership, and corporation in the county, together with the value of such property. The value put upon a property by the assessor is called its *assessed valuation*. A detailed discussion of the methods by which the assessor and his assistants seek to arrive at the true value of the property owned by each individual will probably be out of place, but it should be pointed out that it is comparatively easy to find the value of each piece of real estate (land and buildings), while it is much more difficult to find the value of personal property (furniture, securities, etc.) of each person.

274. The tax rate. — The arithmetic involved in computing a property tax consists of two problems, the first of which is to find the *tax rate*. The total amount of taxes to be raised is decided upon and the total assessed valuation is obtained from the assessor's books. Dividing the total taxes by the total assessed valuation gives the tax rate. The tax rate is expressed variously as so much per $100 worth of property, as so many mills per dollar, or as a certain rate per cent on the value of the property. Suppose the assessed valuation is $2,840,650 and the tax to be raised $49,500. Then 49500 ÷ 2840650 = .01742 which would give a tax rate of $.01742 on the dollar or $1.742 per hundred dollars. In practice the tax rate on the dollar is expressed in mills. Fractions of a mill, such as ½, ¼, or ¾, are often used. Since there is always some loss in collecting taxes, allowance is made for this in fixing the rate. In the case just given the rate might be fixed at 17½ or 17¾ mills per dollar. In many cases there are special taxes whose rates are fixed by law. Thus, there may be a tax of ¼ mill for maintaining a library, a tax of ½ mill to pay interest on certain bonds, and a special tax of 1½ mills for the maintenance of higher educational institutions. The sum of these various rates constitutes the total tax rate for the county or city. A sample tax bill showing all such taxes should be obtained from the county treasurer.

275. Spreading the tax. — The second problem in computing a property tax is to find the amount of tax to be paid by each individual when the tax rate and the assessed valuation are known. If the tax rate is 17½ mills on the dollar, then the tax on a property assessed at $14,800 is .0175 × $14,800 = $259. In the practical work of "spreading" the tax a "tax table" is made for the rate to be used. One form of tax table, of which there are several, gives the tax for every dollar from one dollar to one hundred dollars. Part of such a tax table for the rate 17½ mills is given on the next page. Making a tax table using a local rate will be an interesting and valuable exercise.

Part of Tax Table, Rate 17½ Mills

	$1 to $10	$11 to $20	$21 to $30	$31 to $40	$41 to $50	$51 to $60	$61 to $70	$71 to $80	$81 to $90	$91 to $100
1	.0175	.1925	.3675	.5425	.7175	.8925	1.0675	1.2425	1.4175	1.5925
2	.0350	.2100	.3850	.5600	.7350	.9100	1.0850	1.2600	1.4350	1.6100
3	.0525	.2275	.4025	.5775	.7525	.9275	1.1025	1.2775	1.4525	1.6275

From this table it can be seen not only that the tax on $42 is $0.735, but that the tax on $420 is $7.35, the tax on $4200 is $73.50, etc. To find the tax on $12,630 add the tax on $12,000 to the tax on $630. When definite rates of taxes are levied for different purposes the tax for each rate is often computed separately by making a tax table for each rate. If a bill for taxes is brought in, it may be found to contain several items, each of which was computed separately.

Tax on $12,000 =	$210.00
Tax on $630 =	$11.025
Total tax	$221.025

The writer had the assessor's list for a township brought into a country school and the tax spread as a school exercise. The interest aroused and the checking by having the children work in pairs made it possible to turn out in a very short time an accurate list of taxes levied. Incidentally the children learned much about local government.

It should be made very clear that the arithmetic involved in computing a property tax consists only of the solution of the two problems stated in this and the preceding section. One of these problems is a case in finding the rate in ordinary percentage and the other a case in finding the percentage. The tax table is simply a special device for finding the percentage easily when the same rate is used in many problems.

276. Import duties. — One or two special cases will be sufficient to give the pupils an idea of what is meant by import duties (tariff) and how these duties are collected. A short list of tariff rates actually in force may be obtained and put on the board. Suppose the rate on certain kinds of standard gloves is \$2.50 per dozen pairs (the rate actually in force at the time this is written). It should be explained that this amount is paid to the *customs collector* by the importer at the time the gloves are brought into the United States. The problem of computing the tax for a given quantity of gloves will then be a simple example in multiplication. An import duty fixed at a certain amount for each article without regard to its value is called a *specific duty.*

On other articles, such as woolen goods, the tariff is given as a certain rate per cent of the price at which the goods were bought in the foreign country. The problem of finding the import duty on a given quantity of such goods, when the equivalent of the foreign price in United States money has been determined, is a simple example in percentage. An import duty computed as a certain rate per cent of the cost to the importer is called an *ad valorem duty.*

The teacher may explain if she thinks best how the import value of an article is authoritatively determined for the purpose of computing its ad valorem duty. When the importer buys the goods abroad, a bill of sale is made out which is taken to a United States consul residing in the territory where the goods are bought. He certifies that in his opinion the goods were actually bought at the price named in the bill. This bill of sale must be presented to the customs officials at the point where the goods are brought into the United States.[1]

Many incidental questions are likely to arise: What is smuggling, and how is it prevented? If a traveler brings back things bought abroad, must he pay duty the same as a regular importer?

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[1] Such bills of sale give the value of the goods in terms of the money of the country where the goods were bought. Every three months the Secretary of the Treasury issues a statement giving the equivalent in U. S. money of all kinds of foreign money. By means of this list the value of the goods in U. S. money is found and on this basis the ad valorem duty is computed.

277. Indirect taxes paid in the form of increased prices. — Such taxes as import duties, excise taxes, and license fees are called *indirect taxes*. Such taxes add to the cost of doing business and must be passed on to the final buyer. The increase of the price to the consumer on account of such a tax is nearly always greater than the tax itself. The following is from the business section of a New York newspaper for December 10, 1921.

> One representative (of the clothing industry) pointed out yesterday that the duty (on wool) means all the way from $4.50 to $7 out of the pockets of those who buy the better grades of overcoats. He explained that an overcoat takes four pounds of clean wool which is taxed at 45 cents a pound, making a total of $1.80. If this wool goes through the hands of importer, top maker, spinner, weaver, clothing manufacturer, and retailer at an average profit of 15 per cent, the tax amounts to over $4.50 when it reaches the consumer. Should the manufacturers and retailers charge their ordinary profits the tax would come to over $7.

PROBLEM: The internal revenue (excise tax) on a certain grade of cigars is $3 per thousand. If the cigars are subject to successive profits of 25%, 20%, and 30% at the hands of the manufacturer, wholesaler, and retailer respectively, by how much does the tax increase the final price?

SOLUTION: 1.25 × $3.00 = $3.75; 1.20 × $3.75 = $4.50; 1.30 × $4.50 = $5.85. Hence the price to the consumer is increased by $5.85 because of the tax of $3 levied on the manufacturer.

This is a fair illustration of what happens in a greater or less degree in all cases of indirect taxation.

An indirect tax is very expensive to collect.

REVIEW QUESTIONS

1. Give a list of the purposes for which the money collected as taxes is expended, (a) by a city; (b) by a county; (c) by a state; (d) by the United States Government. (§ 269.)

2. Why are taxes higher now than they were fifty years ago? Are there reasons to believe that taxes are increasing in your community? (§ 270.)

3. What are the main sources from which taxes are collected? (§ 272.)

4. Explain how a direct property tax is levied. What are an assessor's duties? How is the tax rate found? How is a tax "spread"? Explain the construction and use of a "tax table." (§§ 272–275.)

5. What is a direct tax? an indirect tax? Give examples of each.

6. What is meant by "import duty"? How is it collected? (§ 276.)

7. What is the usual relation between an indirect tax and the prices of goods on which such tax is levied? (§ 277.)

CHAPTER XXII

INSURANCE

278. Property insurance. — A contract by which, for a certain consideration called a *premium*, a company agrees to pay a specified sum in the event of the loss of a certain property through the cause stated in the contract is called an *insurance policy*. There is a large variety of losses of property against which insurance may be bought. Loss of buildings and other property by fire; loss of ships and cargoes by the perils of the sea; loss of crops by hail, floods, drought, or insects; loss of trade by bad weather on a certain day; loss of property by war — these and a great many other kinds of losses may be covered by insurance.

The most common kinds of property insurance are marine insurance and insurance against loss by fire.

279. Fire insurance, premium, kind of policy. — An ordinary fire insurance policy is a contract to pay all loss, up to a certain amount, which may arise from the total or partial destruction by fire of a certain specified property. The policy also covers any loss due to the use of water or any other legitimate means used in extinguishing the fire. Thus, if a house which is insured for $10,000 is damaged to the extent of $6,000, the insurance company must pay $6,000; while if the house is damaged to the extent of $12,000, the company pays $10,000.

The amount paid by the insured (the premium) varies with the kind of property insured and with the degree to which it is exposed to fire. The premium is usually computed as a certain number of dollars per thousand or a certain rate per cent of the amount of the policy.

An *ordinary* fire insurance policy specifies the amount for which the company is liable. An *open* policy specifies the rate of premium and the kind of goods insured but does not specify the amount of property insured. Thus all the wheat in an elevator may be insured without stating how much wheat there is in it. The premium is then computed on the actual amount of wheat in the elevator from time to time. The premium on an open policy is not paid in advance since it is not known how much property will be insured and hence the premium is not known.

Companies usually issue a three-year policy for 2½ times the cost of a one-year policy, and a five-year policy for 4 times the cost of a one-year policy.

For local material the rate paid on certain buildings in the neighborhood and on property in these buildings should be obtained; the rate paid on the school building and its contents will be particularly interesting. The rates will be found to vary with the character of the neighborhood, the proximity of other buildings, the character of the buildings, and the uses to which they are put.

280. Reasons for insuring against fire. — It is clear that the total of the premiums paid to insurance companies is considerably greater than the total which they pay to cover losses, for otherwise these companies could not continue in business. Hence the probability is that anyone who insures a property will lose money by so doing. Then why should people insure their property? The reason is this: The payment of a few dollars as premium each year is a slight inconvenience while the destruction of the property insured might cause a very great hardship or even financial ruin. One may without injury draw a certain amount of blood from an animal from time to time so that the aggregate drawn over a long period will be more than the animal possesses at any one time.

The argument for insuring property against fire does not hold in the case of a large concern whose properties are in many different localities. One building belonging to such a company may be destroyed by fire but certainly not many of them. The value of the property which is at all likely to be destroyed is a small matter compared with the company's financial resources or with the total value of the properties on which premiums would have to be paid in case all were insured. If a station of a great railway should be destroyed this would be a comparatively small matter, whereas the destruction of a factory in which a company is doing all its manufacturing might wreck the company. Certain companies of the type described "carry their own insurance." That is, they take out no insurance, while small concerns and private individuals nearly always insure their property against fire.

281. The fire insurance agent. — In every town except the very smallest there are insurance agents who write policies and collect premiums for the more important companies. It is necessary for a company to have an agent in every locality where the company does business. Frequently one agent represents several companies and has some other occupation besides attending to fire insurance. His compensation is always in the form of a commission (see section 215) on the amount of the premiums on the policies which he writes. One of the functions of the agent is to judge the character of the persons who apply for insurance and the degree of risk connected with the insuring of each item of property. In all cities of any size a complete insurance survey is made jointly by all the companies doing business there, and from this survey a description of each building is obtained and the rate which should be paid on it is determined.

Even with such a schedule of rates it is necessary that discretion be exercised in behalf of the company. There are always some people who seek to defraud. One way of doing this is to insure property for more than it is worth and then set fire to it. An insurance company will not knowingly insure a piece of property for more than its value. In this and other respects the agent must look after the interests of the company he represents.

282. The arithmetic of fire insurance. — The actual arithmetic of fire insurance in so far as it is studied in the grades is very simple. The rate and the amount of the policy are given and the pupil is required to find the premium. This is a simple problem in percentage. Problems of a different type may be encountered in a grade course but they can be solved by intelligent use of the equation of percentage (see section 188) provided the child understands the plan according to which the premium is computed. We can not urge too insistently the importance of bringing out the essential similarity of new problems to others that have been solved.

283. Life insurance. — A life insurance policy is a contract in which the insurance company agrees to pay a specified sum on the death of a person or after a certain number of years in case the person insured lives that long. The payment of the amount named in the policy is conditioned on the prompt yearly payment of a specified sum called a *premium.*

284. Different kinds of life insurance policies. — The simplest kind of life insurance policy is the so-called *straight life policy* in which the company agrees, in case the premiums are paid promptly when due, to pay a certain sum at the time of the death of the insured. Under this kind of policy the payments of premiums continue until the death of the insured.

An *endowment policy* is one in which the company agrees to pay a specified sum at the expiration of a certain number of years or on the death of the insured in case that should occur earlier. An endowment policy usually matures in twenty years. That is, in twenty years from the time the policy is written the company will pay the insured the amount of the policy if he is still alive or will pay the amount to his estate or someone named in the policy if he dies earlier. Under this kind of policy the payment of the premiums continues until the maturity of the policy or until the death of the insured if that occurs earlier.

A *limited payment life policy* is payable on the death of the insured but the payments of premium continue only for a limited number of years, such as ten, fifteen, or twenty. Thus in the case of a ten-payment life policy the premiums continue for ten years at which time the policy is fully paid up and no further payments are required.

These are the most important kinds of life insurance policies, but besides these there is an almost endless variety. There are policies providing a pension for the insured when he reaches a certain age or providing a pension for someone depending on him in case of his death. There are joint pensions for two or more people; there are annuities running for a certain number of years, and new kinds or new modifications are appearing constantly.

285. Reasons for insuring one's life. — The reasons why a man having a family depending on him should carry life insurance are very much the same as the reasons for insuring property against fire. These reasons are of particular force in case little or no property has been accumulated. Even those who are rich insure their lives on the ground that a considerable sum in spot cash, available at the time when the manager is gone, may prevent serious sacrifices of property. In the case of a moderately well-to-do family a few thousand in cash may prevent very serious loss and difficult financial circumstances. For a family which has accumulated no property, money from an insurance company is often the only thing that saves a widow and children from dependence on charity when the head of the family dies.

These arguments have had such weight with the American people that the total life insurance in force (the amount which the companies would have to pay if all who are insured were to die at once) is well over $200 per capita or $1,000 per family. The actual amount may be learned by consulting one of the metropolitan newspaper almanacs, a copy of which should always be in the possession of the teacher. They are within the reach of all for they can be purchased for a fraction of a dollar.

Annual Premiums in an Old American Company

Age	Straight Life	20-Payment Life	20-Year Endowment	Age	Straight Life	20-Payment Life	20-Year Endowment
21	19.62	29.84	48.63	41	34.16	43.82	54.60
22	20.06	30.31	48.77	42	35.39	44.90	55.20
23	20.51	30.80	48.96	43	36.70	46.04	55.85
24	20.99	31.31	49.14	44	38.08	47.25	56.56
25	21.49	31.83	49.33	45	39.55	48.52	57.34
26	22.01	32.37	49.53	46	41.12	49.87	58.20
27	22.56	32.94	49.73	47	42.79	51.31	59.14
28	23.14	33.52	49.95	48	44.47	52.83	60.17
29	23.74	34.13	50.18	49	46.46	54.45	61.31
30	24.38	34.76	50.43	50	48.48	56.17	62.25
31	25.05	35.42	50.69	51	50.62	58.01	63.01
32	25.75	36.11	50.96	52	52.91	60.07	64.41
33	26.50	36.82	51.26	53	55.35	62.06	67.05
34	27.28	37.56	51.57	54	57.95	64.29	68.84
35	28.11	38.34	51.91	55	60.72	66.69	70.81
36	28.98	39.15	52.28	56	63.68	69.26	72.97
37	29.90	40.00	52.67	57	66.84	72.01	75.31
38	30.80	40.89	53.10	58	70.22	74.98	77.91
39	31.91	41.81	53.56	59	73.83	78.16	80.73
40	33.01	42.79	54.06	60	77.69	81.60	83.82

This table may be used for making problems in the class.

286. Rates of premiums for different kinds of policies. — The yearly premiums are very different for different kinds of policies.

The straight life policy carries the lowest premium since the payments continue until the death of the insured. It is clear that a limited payment life must carry a higher premium, and the premium is higher the shorter the time that the payments are to continue. Thus a ten-payment life policy carries a higher yearly premium than a fifteen-payment life policy and this in turn a higher premium than a twenty-payment life policy. On the opposite page is a table used by a standard American company showing premiums for $1,000 of insurance. All the "old-line" companies use very nearly the same rates.

287. Reliability of life insurance companies. — The so-called "old-line" life insurance companies are among the safest financial institutions in the world. These companies are required by law to invest their funds in the very safest securities and to accumulate a surplus which experience has found to be quite sufficient to meet all the death losses and other claims which may be made upon it. Moreover it is very much to the interest of all insurance companies that no one company shall fail, because an unshaken general belief in the soundness of insurance companies is necessary in order that they may continue the vast business they are doing. The failure of one would arouse suspicion of all. Hence if a company should be known to be in difficulties, other companies would come to its aid or take over its business so that the insured would not lose. One who insures his life may therefore be certain that he is making an investment which is as safe as any that he can make though he may possibly obtain a very low rate of interest. The longer he lives the lower will be the rate of interest on the money which he pays as premiums. However, this is an investment on which people are not overanxious to secure a very high rate of interest.

288. The arithmetic of life insurance. — The problems on life insurance which should be attempted in a grade course in arithmetic are so simple that no directions or explanations are needed to enable the pupils to solve them.

Example 1. Using the table on page 408 find the yearly premium on a certain kind of policy when the amount of the policy and the age of the insured are known.

Example 2. Under the conditions of Example 1, find the total premiums paid in a given number of years.

These two examples represent the two types that will naturally be solved in the grades. Under exceptional circumstances a bright pupil may solve an example of the following type.

Example 3. Under the conditions of Example 2, what will the total premiums amount to if each premium is invested at 4 per cent interest compounded annually?

To solve this problem a compound interest table must be used and even then it is quite laborious.

It will do no harm to mention that there are many important problems connected with life insurance which are dealt with in special courses in the colleges and which are difficult even then. Insurance companies employ experts, called *actuaries*, whose business it is to solve the many difficult problems that arise.

REVIEW QUESTIONS

1. Define "insurance policy." Against what kinds of losses may insurance be bought? (§ 278.)

2. Describe different kinds of fire insurance policies. How is the premium on such a policy usually computed? (§ 279.)

3. What reasons are there for insuring against fire? (§ 280.)

4. What are the functions of a fire insurance agent? (§ 281.)

5. What problems in arithmetic arise in connection with the subject of fire insurance? (§ 282.)

6. What is a life insurance policy? Describe different kinds of life insurance policies. (§§ 282, 284.)

7. What reasons are there for insuring one's life? Compare rates of premiums on different kinds of life insurance policies. How does the premium differ with the age of the insured? (§§ 285, 286.)

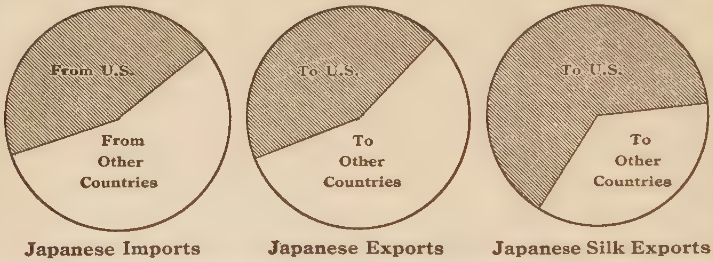
8. Discuss the reliability of insurance companies. Give some of the reasons for their reliability. (§ 287.)

9. Describe the kinds of problems in arithmetic that arise in connection with life insurance. (§ 288.)

CHAPTER XXIII

GRAPHIC REPRESENTATION

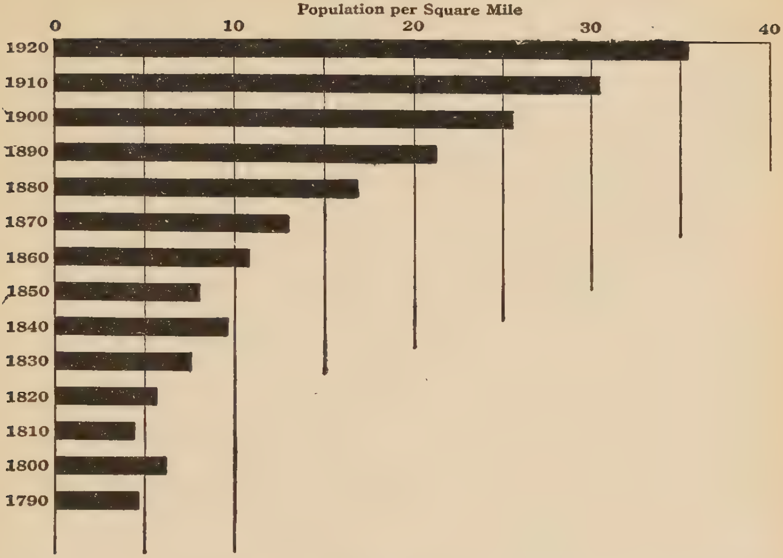
289. The graph and its uses. — Pictorial representations or diagrams of various kinds are used very extensively to represent information which may also be represented in the form of tables. Such representations are called *graphs*. Below are given several examples of graphs culled at random from current magazines and government reports.



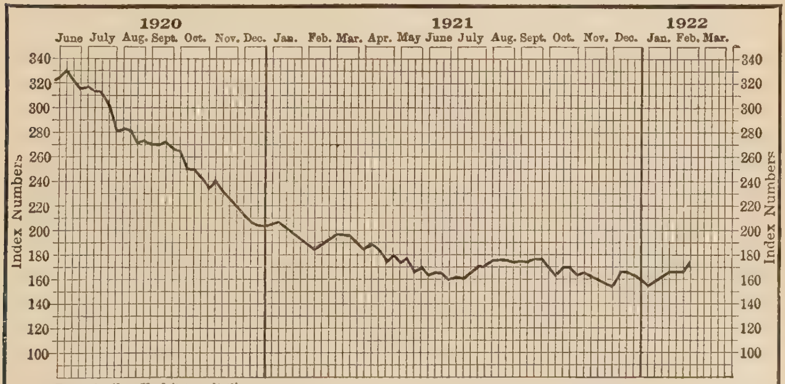
AMERICA'S SHARE IN JAPANESE COMMERCE IN 1920

These graphs are largely self-explanatory. We shall study the two most important types more in detail.

290. The bar graph. — The bar graph is used for a large variety of purposes. Thus, Volume 1 of the 1920 Census contains bar graphs showing the population per square mile in United States for each census from 1790 to 1920 (see below), the population per square mile by states for 1910 and 1920; total population in places (cities) of 8,000 inhabitants or more at each census 1790–1920; urban and rural population 1880–1920; per cent urban in



total population by states 1890–1920; populations in cities having 100,000 inhabitants or more for 1910 and 1920.



From the New York "Annalist."

CURVE OF THE FOOD COST OF LIVING.

This graph shows the fluctuations in the average wholesale price of twenty-five food commodities selected and arranged to represent a family's theoretical food budget.

291. The coördinate or curve graph. — The graph shown on (page 415) exhibits the essential facts in Example 4, section 239.[1] (See page 368.) In this graph each point is determined by its distances from two margins of the graph. Usually two lines are drawn at right angles, and the distances from them to each point is given. The most familiar instance of this kind is the location of a point on the earth's surface by giving its distances and directions from the equator and from the prime meridian, that is, its latitude and longitude. In simple graphs, such as those that should be studied in the grammar schools, we measure in only one direction from each of the two reference lines. That is, we measure *upward* from the horizontal line (or from the lower margin of the graph) and *to the right* from the vertical line (or the left margin of the graph). In the graph shown here each horizontal space represents five years of time, beginning at the age of twenty and continuing to the age of seventy. Each vertical space represents $500, beginning at zero and continuing to $10,500.

As shown in the legend the solid line represents the "earned income," that is, the salary. If this man had saved nothing, the amount actually spent for living expenses would have been the same as the earned income. On retiring at seventy years of age he would have had no property whatever. The total income of the man who saved one third of his total yearly income and invested it at seven per cent is represented by the "dot-dash" line, and the yearly amounts used by him for living expenses is represented by the "dash" line. The dash line is below the solid line from age twenty to age forty-six, where the lines cross. The vertical distance between these lines up to this point represents the difference between the amounts these men would use yearly for

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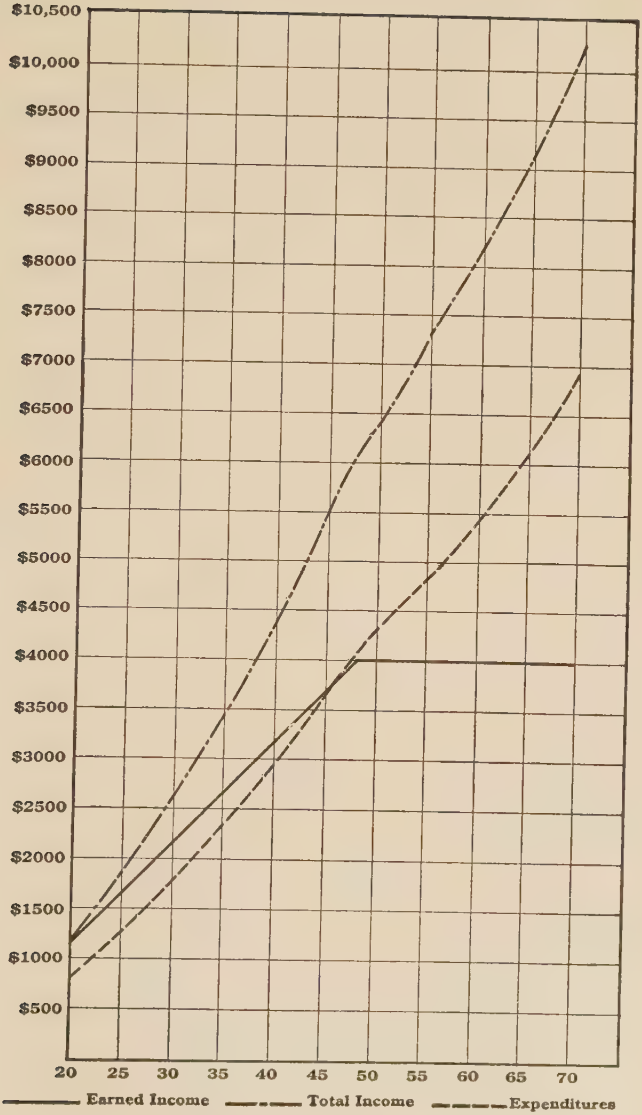
[1] The problem is as follows: At the age of nineteen a boy enters a business establishment at a salary of $1200 a year. He receives a yearly increase of $100 till he reaches $4000, at which salary he continues until he retires at seventy years of age. He saves and invests at seven per cent one third of each year's total income. Find for each year the total income, and the amount available for the year's living expenses.


$10,500
$10,000
$9500
$9000
$8500
$8000
$7500
$7000
$6500
$6000
$5500
$5000
$4500
$4000
$3500
$3000
$2500
$2000
$1500
$1000
$500
20
25
30
35
40
45
50
55
60
65
70
Earned Income
Total Income
Expenditures

living expenses. After reaching the age of forty-six the saver would spend more for living expenses than the nonsaver, in spite of the fact that he continued to save one third of his total income; and as the graph shows, the difference increases very rapidly. When the saver retires at seventy, he has a property amounting to over $93,000 which yields an income of $6,400 a year.

It should be noted that the graphs represent the figures only approximately but the approximation is close enough to convey entirely adequate information for the basing of sound judgment on the value of saving.

292. Material for graphs. — In some papers the temperature at each full hour for the twenty-four hours next preceding is given. These records may be worked up into graphs in the schoolroom. Again average monthly temperatures for some place of interest may be obtained and represented graphically. A little later the pupils will find such graphs in books on physiography or general science. The population of the State at each census since its admission into the Union or of the city since it was founded, the total attendance in the school each year since it was started, etc. may be represented graphically by the pupils. Especially well-made graphs about the growth of the school may be hung in the principal's office for the information of visitors.

A very interesting graph may be made from the data contained in the time table of a railway connecting two cities of interest to the pupils. Let the width of the graph represent the distance between the cities and the height of it represent twenty-four hours. The movement of each passenger train will then be represented by a line; the position of each train will be shown for every hour of the twenty-four; the points of meeting will be indicated, etc. The speed of each train will be indicated roughly, the more nearly horizontal the line the greater the speed.

The material to be used for graphs must be determined largely by the locality; what is interesting and significant in one place may appear entirely far-fetched in another.

293. Practical uses of the graph.— Many practical uses of the graph have already been suggested in this chapter. The uses are almost endless. The engineer uses graphs in innumerable ways: The consumption of fuel, the pressure of steam, the use of electric current, and a host of other things are "graphed" so that the eye may take them in at a glance. The manufacturer represents the various elements of cost by graphs. The merchant has extensive graphs showing monthly and yearly variations of prices. Bankers and brokers have similar graphs showing the variation in the market values of securities. In agricultural experiment stations and in some cases on the farm, elaborate graphs are constructed to show the "performance" of a dairy cow or a herd of cows. The amount of feed consumed, the yield of milk, and the amount of butter fat in it are shown for each day. To these may be added the cost of feed, the value of milk, and other data. The scientist is a most inveterate user of graphs. There are growth curves representing the rate of growth of trees or animals. There are curves representing the amount of heat and light sent out by substances at different temperatures, and so on *ad infinitum.* The advertiser is a firm believer in the "carrying power" of graphs as anyone may see by glancing through the magazines. The statistician uses them at every turn. One can not make intelligent and economical use of the census reports without understanding the meaning of graphs.

294. Figure-blind people. — There are many people to whom figures have little significance. "I can never make anything out of figures," is a common saying. That one number is twice another makes no definite impression on such people. They may be said to be "figure-blind." And still it is true that we are all obliged to adjust ourselves to the quantitative aspects of our environment.

Surely the "figure-blind" are at a great disadvantage unless quantities can be represented in some other way than by figures. That is precisely what the graph does. A person who would experience little beside a sense of confusion by looking at the columns of figures on page 368 might be profoundly impressed by the graph on page 415. Possibly the best way for "figure-blind" persons to learn to appreciate the significance of figures themselves is to actually construct graphs from the figures and then to reflect that the figures contain precisely the same information as the graphs if they only could see it. So much of the information that really matters is quantitative that serious effort to master it in one form or another is decidedly worth while.

295. Importance of the graph in the curriculum. — The importance of studying the graph in the grammar school is measured by the importance of graphs in practical life, and this has been sufficiently indicated. It remains therefore to consider the question as to whether or not the subject is beyond the comprehension of the average grade pupil. On this point there can be little difference of opinion among those who have given the subject serious thought. The construction of a graph, such as one representing the variation of temperature or the yearly attendance at a school, is very simple — much simpler than many other topics that are treated in the grades. School geographies contain graphs the meaning of which the pupil is expected to understand as a matter of course.

It would, I believe, be impossible to defend the exclusion of the graph on rational grounds, even if the time allotted to arithmetic were less than it is now. This subject should certainly be given precedence over all whose exclusion has been seriously advocated and even over many on whose retention in the curriculum there is practical unanimity.

The chief difficulty is that a large number of teachers have had little experience with graphs as a school subject. The introduction of such a topic always encounters opposition. (Compare section 111.) Many teachers are not unduly hospitable in their reception of that which requires the mastery of something new. What they do not already know or what is not apparent in all its details without effort is regarded as far beyond the reach of the pupils. But this is an error. An operation, such as long division, would require for its mastery considerable effort on the part of a mature person of normal intelligence if it were entirely strange to him; and still we teach it successfully in the fourth grade.

However, the graph is bound to make its way into the grammar grades. Its essential simplicity and its wide range of usefulness will make it impossible to keep it out. The graph should be introduced reasonably early in the school course, and the study of it carried through a number of years. It is not sufficient that a few lessons on the graph be given at some one point: it should be developed early enough so that it may be used in a practical way in other school subjects such as geography. That which is only partly mastered in a hasty study will soon be forgotten, and, no matter how simple it may be, it is likely to be considered as something difficult and quite beyond the understanding of ordinary people. The graph should be studied with a view to complete mastery and frequent use.

REVIEW QUESTIONS

1. What is meant by a "graph"? For what purpose are graphs used? (§§ 289, 293.)

2. What is a "bar graph"? Give an example of its use. (§ 290.)

3. Describe a coördinate, or curve, graph. Give an example of its use. Compare the graph on page 415 and the table of figures on page 368. These represent the same facts. Which exhibits the facts more convincingly? (§§ 291, 294.)

4. What material for making graphs can be found in your neighborhood? How may children obtain such material? (§ 292.)

5. Discuss the importance of the graph in the curriculum. In what grades should it be taught? (§ 295.)

CHAPTER XXIV

SQUARE AND CUBE ROOT

296. Two general principles. — There are two distinct methods of finding square root and also of finding cube root. In deciding which of these methods is to be used the following general principles are proposed:

1. *In teaching those operations that are of very frequent occurrence in practical life, the main purpose should be to find the briefest and smoothest forms and to habituate them, the understandability of the forms being of lesser importance.*
2. *In teaching those operations which are of relatively rare occurrence, the main purpose should be to find the forms that are easily understood and remembered, the smoothness and brevity of the forms being of lesser importance. Forms that are not easily understood are always forgotten unless they are used very frequently.*

Obviously these principles, if adopted, will serve to decide which method of solution to use in many cases in which this is still a mooted question. The selection of method for computing interest is a case in point. Unquestionably the easiest method to understand is the *equation method*, "interest = principal × rate × time," sometimes called the *cancellation method;* while the bankers' method (see section 226) in the hands of a good computer is the most elegant, compact, and rapid.

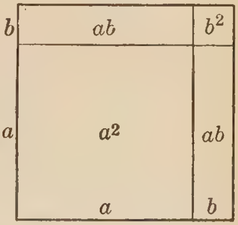
If we use the above principles, the conclusion would be that a person who expects to compute interest at rare intervals should learn the equation method, while he who would compute interest very frequently should learn the bankers' method. We now proceed to examine the two methods of finding square root.

297. First method of finding square root. — The method used most frequently for finding square root is based essentially on the formula,

$$(a+b)^2 = a^2 + 2ab + b^2 = a^2 + (2a+b)b.$$

Since those who are learning to find square root are usually not acquainted with the equation, the equivalent geometric illustration is used.

Before proceeding to the actual solution of an example certain preliminary considerations are necessary: Since $10^2 = 100$, it follows that the square root of any number between 0 and 100 lies between 0 and 10. Again, since $100^2 = 10000$, it follows that the square root of any number between 100 and 10000 lies between 10 and 100. From such considerations follows the rule for dividing a number whose square root is to be found into groups of two figures in each, beginning at the decimal point.



Example: Find the square root of 5493 to the nearest integer.

Solution: If we divide the number into groups as described above there will be two groups, the left one of which is 54. The largest square contained in 54 is 49 whose square root is 7. Subtract 49 from 54 and bring down the next group (93) obtaining 593. Now add one zero to the root already found and multiply by 2, obtaining 140 which is a "trial" divisor. 140 is contained 4 times in 549. Replace the 0 in 140 by 4, multiply by 4, and subtract from 593. This process is made clear by referring to the diagram above. The 7, which is really 70, is the *a* of the figure. To get the rest of the figure we multiply twice *a* and also *b* by *b*. But 144 is precisely twice 70, plus 4. There is still one question left unanswered, viz: How do we know that 74 and not 75 is the *nearest* approximation to the required root?

$$\begin{array}{rr} & 74. \\ & \overline{)54\ 93} \\ & \underline{49} \\ 140 & 593 \\ 144 & \underline{576} \\ & 17 \end{array}$$

To carry the approximation further and to explain the process by means of the diagram it would be necessary to regard the part of the root already obtained, 74, as the *a* of the figure and to point out that 74^2 has already been subtracted so that the next step would be to double 74 and annex a zero to obtain the next trial divisor. In practice the explanation by means of the diagram usually stops when two figures are obtained, the subsequent steps being taken by analogy.

298. Second method for finding square root. — The preliminary work leading to the division of the number into groups is exactly the same as in the first method.

We now recall that the square root of a number is one of its two equal factors. To find the square root of 5493 we notice that the square root of 54 lies between 7 and 8 and that hence the root of 5493 lies between 70 and 80. Suppose we estimate 75 as an approximation to the root. Dividing 5493 by 75 we get 73.24 as the quotient. That is, the root lies between 75 and 73.24. It is clear that 74 is the nearest approximation in integers.

To make this method clear we solve one more problem:

EXAMPLE: Find the square root of 43714, correct to four places of decimals.[1]

$$\begin{array}{r} 208.16 \\ 210\overline{)43714} \end{array} \qquad \begin{array}{r} 209.0778 \\ 209.08\overline{)43714} \end{array} \qquad \begin{array}{r} 209.07896 \\ 209.0789\overline{)43714} \end{array}$$

SOLUTION: It is evident that the square root of 4 37 14 lies between 200 and 300. Moreover it is very much closer to 200 than 300. We estimate the root to be 210. Dividing 43714 by 210 we get 208.16 as the quotient. Now take half the sum of 210 and 208.16, or 209.08, as the next divisor, and we get 209.9778 as the quotient. Taking half the sum of 209.08 and 209.0778, or 209.0789, as the next divisor we get 209.07896. Hence we know that 209.0789 is the nearest approximation to four places. Indeed we know that 209.07893 is the nearest approximation to five places.

The following rule describes this method:

1. *Find a first approximation.*
2. *Divide the number by the first approximation.*

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[1] In practice numbers that are exact squares occur so seldom as to be almost negligible. Hence the roots are approximations to a certain number of decimals.

3. *Take half the sum of the first approximation and the first quotient as a second approximation, and divide the number by it.*
4. *Take half the sum of the second approximation and the second quotient as a third approximation and divide the number by it.*
5. *Continue this process as long as may be necessary to obtain the required closeness of approximation.*

The root always lies between the last divisor and the last quotient. The approximation is accurate to at least as many decimals as the last divisor and quotient are alike.

If we apply the principles of page 420, the selection of this second method for use in the grades is inevitable.[1] Occasions in practical life requiring the finding of square root are comparatively rare. The first method (see section 297) is difficult to understand, especially when there are more than two figures in the root; the formal rule is sure to be forgotten; and in case of real need the individual is helpless. The second method is very easily understood. The writer knows from many experiences that it can be taught in fifteen minutes so that an average adult of limited education will use it fairly effectively. The only trouble is in making a fairly close first approximation, though this is not essential to success but only makes a little difference in the length of the work. Suppose we are to find the square root of 942638. Suppose 900, a very poor first approximation, is used. Dividing and finding the second approximation we have 973, which would be a very good first approximation. It should be borne in mind that the successive approximations obtained by this method are generally a little too large. This may be illustrated by applying the method to a simple case to which it would never be applied in practice. Suppose 2 is taken as the first approximation of the square root of 16. Then the first quotient is 8, and the second approximation is 5, which is one larger than the square root of 16. Suppose 5 were taken as the first approximation of the square root of 16 then the quotient would be 3.2 and the second approximation 4.1; if we continue, the third approximation would be 4.0012 and the fourth approximation 4.00000018. This example also illustrates how rapidly the root is approximated; the last approximation in this case being correct to six places of decimals. Obviously the square root of 2 or of 3 could be approximated just as easily and accurately.

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[1] "Yet even square root is probably best made more closely a part of the total ability, being taught as a special case of dividing where divisor is to be the same as quotient, the process being one of estimating and correcting." Thorndike, *op. cit.* (p. 42), p. 138.

299. The finding of cube roots. — While it is not recommended that cube root be taught in the grades, it is believed that the extension of the method of the preceding section to cube root is of sufficient interest to warrant a brief consideration of it here. By definition, the cube root of a number is one of its three equal factors.

Example: Find the cube root of 29734 correct to two places of decimals.

$$\begin{array}{r} 30.94 \\ 961\overline{)29734} \end{array}$$

Solution: Dividing into groups of three figures, as 29 734, we see at once that the cube root lies between 30 and 40 since $30^3 = 27,000$ and $40^3 = 64,000$. Moreover the root is much nearer 30 than 40. We take 31 as the first approximation. Now square 31 obtaining 961. Dividing 29734 by 961 we get 30.94. As our second approximation we take one third the sum of the three factors 31, 31, 30.94, or 30.98 which is the required approximation. To verify this, square 30.98 and divide the result into 29734 obtaining 30.98063, which also gives 30.98021 as another approximation which is correct to five places of decimals.

Carrying the approximation of this root to seven significant figures (as has just been done) by the usual formula method is a very laborious operation.

To approximate the cube root of a number by the method just illustrated, use the following rule:

1. *Estimate a first approximation, and divide the number by the square of this approximation.*
2. *To the quotient thus obtained, add twice the first approximation, and divide the sum by 3, thus obtaining a second approximation.*
3. *Continue this process as far as may be necessary to obtain the required closeness of approximation.*

The root always lies between the last quotient and the next preceding approximation. The approximation is accurate to at least as many decimal places as the last quotient and the next preceding approximation are alike.

300. Importance of square and cube root. — In practical work square root occurs with sufficient frequency to justify some attention to it, especially if a simple, obvious method such as the one used in section 298 is used. Cube root is needed so seldom that no considerable time should ever be devoted to it in the grades. If the usual method is used, there would seem to be no defense for it whatever. If the method of section 299 is used, it may be introduced as an interesting extension of square root. This will require very little time, and the method once learned will never be forgotten while the other method is almost universally forgotten.

REVIEW QUESTIONS

1. State in substance the two general principles given in section 296.

2. Describe the two methods for finding square root. Just how much of the work is common to the two methods? Which of the two methods is more easily learned and remembered? (§§ 297, 298.)

3. Describe the method given in section 299 for finding cube root. What reasons may be given for teaching cube root in the grades if taught by the method suggested here? if taught by the usual formula method? Discuss the importance of square and cube root. (§ 300.)

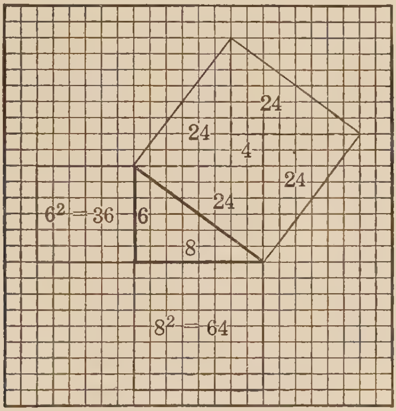
CHAPTER XXV

MENSURATION

301. Measurement and mensuration. — "Measurement" has already been defined (page 278) as "the process of dividing the quantity to be measured into parts each equal to the unit of measure and then counting these parts." Measurement is a *direct* process in which the unit of measure is applied directly to the quantity to be measured. The length of a straight line is usually, though by no means always, measured in this way. On the other hand the area of a rectangle is very seldom measured by a direct application to it of the unit of area. The area of a floor is never found by applying to it the unit of measure, a square foot. The length and the width are found by direct measurement and the area is computed according to a known rule. Similarly the cubic contents of a bin is found by measuring the length, width, and depth and then applying the formula, "volume = length × width × depth." The area and the volume are found by *indirect measurement.* The science which deals with indirect measurement is called *mensuration.*

Among the magnitudes whose indirect measurement, or mensuration, is considered in the grades are straight lines (in some cases), areas of rectangles, triangles, trapezoids, and circles; volumes of rectangular solids, prisms, cylinders, pyramids, cones, and spheres; surfaces of cylinders, cones, and spheres, as well as surfaces of other solids whose surfaces consist of plane figures.

302. Indirect measurement of straight lines. — The only practical means taught in the grades for measuring a straight line indirectly are the Pythagorean proposition and the use of similar triangles. That the square on the hypotenuse of a right triangle is equal to the sum of the squares on the two sides (the Pythagorean proposition) may be verified by actual measurements. Thus, if a right triangle is constructed whose sides are 6 inches and 8 inches it will be found that the hypotenuse is 10 inches ($10^2 = 6^2 + 8^2$). A careful construction on squared paper will



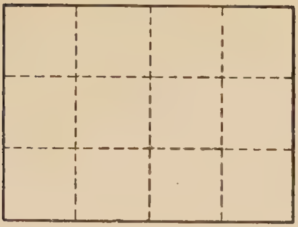
verify this proposition for all simple cases. Aside from the fundamental importance in more advanced mathematics the Pythagorean proposition is of some practical importance in the trades and industries. The carpenter uses it to place two timbers at right angles using lengths 3, 4, and 5; or 6, 8, and 10 as the sides of a triangle. Usually he knows one special case such as that, if the sides of a triangle are 6, 8, and 10, it has one right angle; but he knows nothing about the general theorem.

The other method used for measuring lines indirectly is based on the use of similar triangles. An adequate development will be found in most modern texts. Very little practical use is made of this method, but indirectly all methods used by the surveyor, geographer, and astronomer for finding the lengths of lines that can not be measured directly are based on similar triangles. The pupils are greatly interested, however, in finding the height of a building or the distance across a lake or a stream without actually measuring it. The proudest moment of the writer's intellectual career was when he devised a method for finding the distance across a lake while remaining on one side of it. He did not care to know the width of that lake nor did he attach any importance to the result when found (it was probably wrong by twenty-five per cent), but *to do it* seemed a miracle and to know *how* it could be done was as enthralling as is the smartest card trick of the parlor favorite to the admiring group about him.

No little keen interest may be aroused in man's achievements by pointing out items of information which he has obtained through indirect measurement of straight lines. By this means he has learned the height of mountains that have never been scaled and of distances between inaccessible peaks. He has calculated the distances to the moon, the sun, and to many of the fixed stars. He knows the size of the earth, the moon, the sun, and the planets though none of these have been measured directly. By means of certain devices that are the product of the human brain, things have been done which would be utterly impossible to mere physical prowess.

303. Mensuration in the lower grades. — It is not our purpose to give here a detailed discussion of all of that part of mensuration taught in the grades. Many texts give excellent treatments of these topics and by consulting a number of them, such as should always be found on her desk, the teacher will find material for as good explanations as can be made. We shall, however, revert to a matter of fundamental importance in considering the area of a rectangle and the volume of a rectangular solid.

In learning the multiplication table children often draw rectangles and divide them into squares to illustrate the products. Thus 3 × 4 or 4 × 3 is illustrated by the figure. From this the



child learns that if he multiplies the length of the rectangle by its width the product is the number of squares contained in it. If the sides are measured in inches, the squares are square inches; and if the sides are measured in feet, the squares are square feet. If such illustrations are used freely in connection with the tables, the idea of the area of a rectangle will be fully developed by the time the multiplication table is learned. As a review it may now be pointed out that the length of the rectangle shows how many squares may be placed in one row along a side, while the width shows how many such rows there are. Hence the product of the length and the width gives the number of unit squares in the rectangle. The following rule will now be fully comprehended:

If the number of units in the length of a rectangle is multiplied by the number of units in its width, the product is the number of square units contained in it.

It will be understood fully also that the sides of the unit square will be equal to the unit used in measuring the length and width. For practical use this rule is contracted into the equation,

$$\textit{area} = \textit{length} \times \textit{width}.$$

When this equation is used, it is understood that what is really meant is the rule just stated. Concise expressions of this kind are of greater importance than many suppose. They are easily remembered; and if once understood, there is little danger that they will lead to haziness. Moreover, they are convenient and economical instruments of thought which enable the mind to do by means of them what it could not do by using the equivalent but longer detailed statements. The history of mathematics is replete with instances in which convenient expressions have set the mind free to work out far-reaching results which otherwise would have been quite impossible of attainment. The Arabic (Hindu) system of numerals is an important case in point. One of the indispensable conditions for intellectual progress is the representation of complex ideas by means of compact expressions. It would be easy to show that by far the greater part of even a very modest speaking vocabulary has come into existence for this very reason. Instead of "a building where children go to study and recite lessons," we say "schoolhouse;" and instead of "a wagon which is moved by an engine carried on the wagon etc.," we say "automobile." It requires self-restraint to write temperately about the custom, all too prevalent in our schools, of carrying so-called "analysis" to an extreme and of preventing the pupils from using short and economical expressions as soon as their meanings have been mastered.

Similar remarks apply to the volume of rectangular solids. By means of unit cubes placed in a box or by means of pictures, it is made clear that the number of cubes that can be placed in one layer on the bottom of the box is found by multiplying the length by the width. The number of units in the depth of the box shows the number of such layers which can be placed in it. The relation between the units of length and of volume will be worked out as in the case of areas. The rule for finding the volume is then stated (compare rule for area, page 429) and is finally contracted into the form:

$$\textit{volume} = \textit{length} \times \textit{width} \times \textit{depth}.$$

This concise rule should then be used in the solution of problems without further analysis. If it is desired to examine again the precise meaning of the rule, this may be done at any time as a review but should not be done in connection with problems in which the rule is used. The pupil should be led to think in terms of such simple statements and to work with them as his efficient tools. Similar remarks apply to all the equations in section 188.

304. Topics of practical importance in mensuration. — Besides the area of the rectangle, just considered, the areas of the triangle and the trapezoid are important especially for the farmer who needs to measure pieces of land.[1] The trapezoid is encountered often, and it should be emphasized that two of its sides must be parallel. In actual practice pieces of land are often measured incorrectly. In such cases the area obtained is nearly always too large. The usual rule for finding the area of a triangle is not easy to apply accurately in practice, since it is difficult to measure the altitude accurately. For this reason it may be worth while, especially in rural schools, to teach the rule for finding the area of a triangle when its sides are given. If the sides of a triangle are a, b, and c and half of the sum of the sides is s, then the formula for finding its area is:

$$area = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}.$$

The area of the circle is important because it is needed in finding the volume of a cylinder.

Besides the volume of a rectangular solid, the volume of a cylinder is the only one of considerable importance. The volume of a pyramid and of its frustum may occur in such problems as finding the volume of a concrete fence post that tapers towards the top. In some cases such a post is not in the shape of a frustum and the problem of finding its volume is somewhat complicated.

The finding of the surface or the volume of a cone or a sphere occurs very seldom in the practical work of one whose mathematical education has been confined to the arithmetic of the grades. The lateral area of a cylinder occurs not infrequently and with it the circumference of the circle. Such problems as finding the lateral area of a pyramid are of very little practical use.

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[1] The surveyor has of course constant need of the rules for finding areas of such figures, but his education is more elaborate than that supplied by the grade schools.

REVIEW QUESTIONS

1. Distinguish between measurement and mensuration.

2. Describe various methods for measuring a straight line indirectly. (§ 302.)

3. What topics in mensuration should be taught in the lower grades? What part is usually introduced in connection with the study of the multiplication table? (§ 303.)

4. What topics in mensuration are particularly important? How does the answer to this question depend upon the location of the school? (§ 304.)

CHAPTER XXVI

THE ARITHMETIC OF THE HOME

305. The home as a business institution. — Conducting the affairs of a home involves a large variety of problems of a business or technical nature which require arithmetic for their solution. There are obvious simple problems, such as computing the amount of a purchase, finding the number of yards of goods needed for certain garments, verifying the reading of a gas meter, making out checks, and keeping a correct record of one's balance in the bank. Ample preparation for the solution of these is given in the ordinary course in arithmetic. There are, however, problems of much greater practical importance to which little or no attention is usually given. What is the real cost of keeping a servant or of owning an automobile? What is the yearly cost of owning or furnishing a house? Given the prices of different articles of food as they vary from month to month, what foods shall be purchased so as to secure a proper diet for the smallest possible outlay? Records of expenditures should be kept to form a basis for intelligent discussion as to whether or not these are being wisely made. Proper use of the information obtained from the solving of these and other similar problems will make a very considerable difference in the economic welfare of a family, especially one with a moderate income.

The problems in question are largely problems of expenditure; and since the major part of a family's expenditure is made by the housewife, they are largely women's problems. To the girls in a class, probably eighty per cent of whom will be charged with the responsibility of a home, there are no problems in arithmetic of even approximately as great practical importance as these.

If, by intelligent consideration of the various problems involved in making the most effective expenditure of the family's income, even five or ten per cent can be saved and properly invested, this will amount to a substantial sum in the course of a lifetime.

But all these problems are not exclusively for girls. The cost of running a car or owning a house will naturally be considered by men at least as much as by women (if considered at all). Moreover, those problems that naturally come within the province of the wife are much more likely to receive adequate and long-continued attention if the husband knows enough about them to appreciate their importance. Of the six years or more in which arithmetic is now studied in the grades, it would seem that one or two months might reasonably be devoted to the study of problems of the type considered in the remaining sections of this chapter.

306. The cost of owning a car. — A concrete example will serve to show the nature of this problem.

A car is bought for $1200 and sold at the end of 3 years for $500. It averages 14 miles on a gallon of gasoline and 200 miles on a gallon of oil. Repairs for three years amount to $150, and insurance to $45. During the three years 8 tires are bought at $30 each; gasoline costs 26 cents a gallon, and oil, $1.10 a gallon. Find the yearly cost if the car is driven a total of 18,000 miles and if interest on the original cost is figured at 6%. If the total income of the family is $3600 yearly, what per cent of this income goes for the car?

Preparatory to the solution of this complete problem, similar problems should be split up into simpler ones such as:

If a car runs 14 miles on a gallon of gasoline costing 26 cents a gallon, what is the cost of the gasoline required to run it 18,000 miles?

Surely by the time he leaves the grades, the child should be able to solve the complete problem as stated above.

It is not within our province as teachers to assert that a family is or is not justified in spending, for instance, one sixth of the total income for the use of a car, but it is decidedly within our province to direct attention to the problem, to guide in its solution, and, what is really more important, to develop, if possible, an attitude on the part of the pupil which will make it likely that, when in mature life a problem of this type does arise, he will actually solve it and consider the information contained in the solution. We are not in a position to quarrel with a person about his preferences, but we are entitled to the opinion that it is not sensible for him to act in the dark. Light is said to be one of the aims of education.

307. Cost of owning or furnishing a house. Cost of a servant. — Again we will give concrete examples:

Example 1. A frame house costs $4500. Yearly depreciation is 4% of the original cost, repairs $60 a year, taxes $45, and insurance $25 a year. Find the yearly cost if interest on the original cost is 5½%. To what monthly rental is this amount equivalent?

Example 2. By buying a certain type of furniture a house can be furnished completely for $1400, and by buying a very much better type it can be furnished for $3500. The better grade of furniture depreciates 8⅓% each year and the other 12½% each year. If interest on the original investment is 6%, what is the difference in the yearly cost of furnishing the house with these two types of furniture? How much money must be invested at 6% so that interest shall amount to this difference?

Example 3. The wages of a servant are $10 a week. The cost of food is estimated at $4 a week and extra waste in the kitchen at $10 a month. By hiring the servant $3.50 a week for day work is saved. What is the net yearly cost of keeping the servant? What per cent of a total income of $4200 is this?

308. The cost of food. — Subject to the general condition that from ten to fifteen per cent of the food served should consist of protein and to certain other minor requirements, the housewife is at liberty to suit the fancies of her household in her choice of foods. The food value of a substance is measured in terms of a unit called a *calorie*. There are tables published in government reports and elsewhere giving the number of calories per pound or per quart of all food substances in common use. Such a table can be obtained at slight expense. The housekeeper's problem is to buy food at the smallest cost per calorie and at the same time maintain proper diversity of diet and the proper balance as to proteins and other food substances. The table of food values will tell her for instance that one pound of porterhouse steak contains 1077 calories; one pound of roast beef, 1560 calories; one pound of round steak, 867 calories; one pound of shoulder, 700 calories; one pound of sirloin, 957 calories. The prices of these will vary so that one week it may be cheaper to buy roast beef; another week, round steak; and still another week it may be cheaper to buy sirloin.

Example: Using the calorie values just given of different kinds of beef, find the costs per thousand calories at the following prices per pound: porterhouse, 30 cents; roast beef, 27 cents; round steak, 18 cents; shoulder, 16 cents; sirloin, 24 cents.

The results in this case are: porterhouse, 28 cents; roast, 17 cents; round steak, 21 cents; shoulder, 23 cents; sirloin, 25 cents. It will thus be seen that by buying a roast instead of a porterhouse, there is a saving of 11 cents out of 28 cents, or over 39 per cent, a fact of which the housekeeper would be in complete ignorance unless she made the necessary computations. Next week these computations may have lost their value inasmuch as the prices may have changed, and hence they must be performed over again. When we consider the large number of food substances that are in daily use and the many possible choices that should depend upon the price per calorie, we appreciate the very considerable amount of computing necessary to buy food effectively.

Halibut steak is 25 cents a pound and whitefish 28 cents a pound. Which is cheaper? Lamb chops are 32 cents a pound; lamb loin, 30 cents; leg of mutton, 25 cents a pound; and pork chops, 28 cents. Which is the cheapest? Without performing the actual computations it is practically impossible to answer such questions.

In a laborer's family $40 a month is used for food. If by proper buying the wife can save $6.00, this may be equal to the amount her husband earns in fifteen hours of work. An hour a week of intelligent thought and computation will surely repay her for her work. All that she needs is to make intelligent use of the right kind of elementary education.

309. Record of expenditures. — On page 438 is shown the record of expenditures kept by a family in moderate circumstances. In the real record book the pages are about three times as wide as the page of this book and there are many columns which are omitted here. There is one page for each month and one line for each day. When the record for the month has been finished each column is added and the total transferred to a page on which one line is given to each month. The totals of these columns are the amounts spent during the year for the various purposes.

After a family has kept such a record for a number of years, the members of the household will be able to see how their tastes have changed and to consider whether or not they are spending their money more or less wisely as the years go by.

The problem of so apportioning a certain income among the various uses to which it may be put as to produce the greatest possible satisfaction and welfare for all members of the family is a very difficult one. It is only by reflecting on the figures obtained from records of expenditures that this problem can be solved even approximately. Experience has proved that the great majority of housekeepers will not keep such records for any length of time. Many start out bravely enough, but the task of footing the columns proves irksome, and in the course of a longer or shorter time all but a few quit it. If the real money value of such records were understood and if sufficient drill to make the work easy had

DAILY EXPENDITURES

Month January

	CLOTHING WIFE	CLOTHING HUSBAND	CLOTHING CHILDREN	MEAT AND FISH	GROCERIES	TRANSPORTATION	BOOKS PAPERS	MISC.
1				$ .41	$ .49			$ .35
2	$ .40	$ .50	$ .60	.35	.63		$ .30	2.80
3				.85	.87	$1.20		
4								.50
5	3.50		1.35	.45	.90			
6				.42	.44	.10		
7				.36	.73			1.60
8		2.60		.94	.29			
9					.34			
10	4.20		.75	.47	.90	.20		
11								
12				.36	.45	.10	.04	
13				.39	.27			.90
14		1.50		.73	.38			.47
15					.74	.20	.02	.25
16			1.00	.45	.97			
17				.27	.90	.20		
18						.25		.20
19	8.60			.46	.47			.20
20				.64				
21				.37	.61	.20	.04	
22		5.00		.29	.34		.03	.75
23				.87	.81			.30
24				.53	.38	.10	.05	.50
25						.15		
26				.12	.67	.25		
27				.42	.76	.10	.10	3.60
28	3.10			.19	.28			
29		3.20		.34	.43		.50	
30				.63	.84	.20	.03	.25
31								
Totals	19.80	12.80	3.70	11.31	14.89	3.25	1.11	12.67

MONTHLY EXPENDITURES

MONTH	CLOTHING WIFE	CLOTHING HUSBAND	CLOTHING CHILDREN	MEAT AND FISH	GROCERIES	TRANSPORTATION	BOOKS PAPERS	MISC.
Jan.	$19.80	$12.80	$3.70	$11.31	$14.89	$ 3.25	$1.11	$12.67
Feb.	4.20	3.60	9.00	10.60	16.39	1.65	.60	28.06
Mar.	3.65	5.89	6.00	10.80	13.41	1.60	3.42	7.12
Apr.	32.45	18.00	2.35	9.65	12.81	2.30	.57	6.80
May	4.85	.75	18.40	9.90	11.46	1.45	9.24	31.40
June	3.65	25.00	7.70	8.60	14.84	2.80	3.41	6.60
July	11.70	4.00	1.25	8.40	13.65	2.60	3.61	8.90
Aug.	8.90	12.60	3.95	7.35	13.80	15.30	.50	70.60
Sept.	26.60	1.75	12.05	7.50	11.40	2.60	3.62	5.30
Oct.	10.00	25.00	1.99	7.70	12.65	1.40	3.45	5.40
Nov.	1.50	4.60	5.90	8.40	19.90	10.70	2.26	3.67
Dec.	4.50	5.60	4.30	9.20	14.60	1.90	1.39	89.65
Totals	131.80	119.59	76.59	109.41	169.80	47.55	33.18	276.17

YEARLY EXPENDITURES

YEAR	CLOTHING WIFE	CLOTHING HUSBAND	CLOTHING CHILDREN	MEAT AND FISH	GROCERIES	TRANSPORTATION	BOOKS PAPERS	MISC.
1918	$131.80	$119.59	$ 76.59	$109.41	$169.80	$ 47.55	$ 33.18	$276.17
1919	152.35	131.60	103.30	121.70	188.45	121.61	40.80	361.40
1920	145.65	132.60	116.70	124.50	186.40	104.79	57.30	317.60
1921	149.60	121.40	137.60	114.20	164.60	89.37	49.50	297.40
1922	127.50	97.60	145.93	119.83	172.74	191.60	64.20	385.67

been given in the schools, no doubt a much larger number would go on with the records until their value would begin to appear. We talk a great deal about making the schools practical, but we have largely avoided anything that is at all laborious or which requires persistent attention and thought. We have been controlled in no small degree by the pedagogy of easy tasks.

REVIEW QUESTIONS

1. Discuss the home as a business institution. How may intelligent use of arithmetic increase the efficiency of the housekeeper? (§ 305.)

2. What elements must be considered in order to determine the real cost of keeping a car? (§ 306.)

3. What information of value may a housekeeper obtain by solving problems like those in section 307?

4. What use of arithmetic must a housekeeper make in order to buy food economically? (§ 308.)

5. Discuss the value of keeping a complete record of family expenditures. How does the value of such records increase if they are kept over a period of years? (§ 309.)

CHAPTER XXVII

ARITHMETIC FOR THE FARM

310. Need of arithmetic for the farm. — The farmer, if he does his work well, needs to use arithmetic more frequently than does the carpenter, bricklayer, skilled mechanic, clergyman, doctor, lawyer, elementary teacher except the teacher of arithmetic, or the teacher in the higher and secondary schools except the teacher of mathematics or exact science. He requires a more diversified use of it than the vast majority of those engaged in trading, transportation, or even banking. The truth of this statement is borne out even by the very limited list of farm problems given in this chapter. When we consider further that the number of people engaged in farming is several times as great as the number engaged in any other single occupation and that many of the farmer's problems are quite different from those that occur in commercial work, the need for special farm arithmetic becomes apparent.

Our early arithmetics had their origin in the needs of the trader, the banker, and men engaged in other types of business. They were essentially business arithmetics. In the course of time these books were adjusted more and more carefully to the powers and interests of the immature learner, but it must be freely confessed that they still retain very largely the characteristics of the earlier avowedly commercial types. If a series of texts were written with the farmer's needs as the center of gravity, they would be quite different from any books now used as basal texts anywhere in the United States.

Much of the arithmetic used by the farmer is commercial; he needs to figure interest, commission, and discounts. There are numerous farm problems which are solved by obvious application of the ordinary processes taught in the schools. But there are many farm problems involving arithmetic the solution of which is not obvious to one with limited training or intellectual penetration. There are many situations in which arithmetic can be used with great profit where the existence of a problem is not suspected by the vast majority of farmers, and in many cases the problem can be solved only when one has possession of special information such as is supplied in courses on agriculture. In this chapter it is proposed to state a series of farm problems and to indicate methods for solving some of them.

311. Cost of feeding farm animals. — The following example illustrates a type of problems which may frequently be solved with profit by the farmer:

Example: A farmer has in storage ear corn, oats, alfalfa, and timothy hay. He learns from a U. S. *Farmers' Bulletin* that each of the following daily rations will serve a horse equally well:

1st Ration		2nd Ration	
Ear corn	10 lb.	Oats	8 lb.
Alfalfa hay	5 lb.	Alfalfa hay	4 lb.
Timothy hay	5 lb.	Timothy hay	6 lb.

The farm[1] values of these are: ear corn, 68 cents a bushel; oats, 41 cents a bushel; alfalfa, $18.50 a ton; timothy hay $16.50. (One bushel of ear corn weighs 70 lb.; and one bushel of oats, 32 lb.) Which ration is the cheaper? How much will a farmer save by feeding the cheaper ration to 6 horses for 30 days (one month)?

In practice a farmer may need to solve such problems many times if he is to feed his horses economically since the relative market values of the different feeds is constantly changing.

Practical Work: The following are standard rations for feeding 1000-pound steers:

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[1] Farm value is the market value less the cost of marketing.

1st Ration		2nd Ration		3rd Ration	
Straw	5 lb.	Stover	10 lb.	Straw	8 lb.
Silage	18 lb.	Silage	15 lb.	Legume hay	6 lb.
Ear corn	12 lb.	Ear corn	12 lb.	Cottonseed cake	5 lb.

Find the local values of these feeds and determine the total cost of feeding steers having a total weight of 20,000 pounds for 30 days, using each ration.

This work can now be extended by obtaining *Farmers' Bulletins* giving standard rations for other kinds of animals and computing their relative costs on the basis of local prices. In some cases as many as a dozen different equivalent rations will be given.

312. Amount of labor required in farming operations. — The relative amounts of labor required for different farming operations have been studied with care. The work required in raising and cutting once one acre of hay has been taken as a unit of man labor and also as a unit of horse labor. The time required to do one unit of farm work depends upon the effectiveness of the worker, the size and convenience of the farm, the effectiveness of the management, the kind of machinery used, etc.

On a well-organized farm 300 units of work may be done by one man in a year and on some farms it may run as high as 400. With very rapid work and extensive methods of farming a man may do one unit of work in five hours, while with intensive methods as high as twenty hours may be required. With average methods and average efficiency ten hours are usually required. The number of horse-hours of work required is usually somewhat greater.

Example: On a certain farm the total work to be done in one year is represented by the following: 12 acres of alfalfa cut 3 times, 24 acres of hay cut once, 20 acres of corn, 20 acres of wheat, 15 acres of oats, 12 dairy cows, 20 head of other cattle, 5 brood sows, 38 other hogs, and 50 hens. Find the total number of units of work required on this farm using the following standards: One acre of hay cut once, one unit; an acre of hay cut 3 times, 3 units; one acre of corn, 6 units man work and 6 units horse work; one acre of wheat or oats, 2 units man work and 3 units horse work;

one dairy cow, 15 units man work and 2 units horse work; 10 head of other cattle, 20 units man work and one unit horse work; 10 brood sows, 30 units man work and 5 units horse work; 50 hogs, not brood sows, 25 units man work and 5 units horse work; 100 hens, 15 units man work and 2 units horse work.

After solving such a problem the farmer has a basis for judging the effectiveness of his management and type of farming since he then knows how many units of work each man and each horse does in one year.

Practical Work: Obtain a standard list of the number of work units required for various farm operations and solve problems similar to the above for farms in the neighborhood.

313. Farm inventories. — One of the simplest and most important of all the applications of arithmetic to the problems of the farmer is the taking of farm inventories. This is fundamental in all methods of farm accounting, and hence anyone who keeps any sort of farm accounts worthy of the name must make up an inventory each year. Even if no other farm records or accounts are kept, an inventory has many elements of value:

1. A complete list of all items of property with their values shows the farmer his financial condition as nothing else can. There is a tendency to feel rich when there is money in the bank, but this is often misleading. One year a farmer's inventory may show $18,500 farm property with little or no cash and a debt of $500 at the bank. The next year his inventory may show a total of $16,000 in farm property with $1500 in cash in the bank and no debt. Clearly he is $500 poorer the second year though very likely he may feel richer. Only a complete inventory will reveal the true state of affairs.

2. There is a tendency to lose small items of farm property, such as tools, through lending or sheer neglect. When the yearly inventory is taken each item in the preceding inventory must be accounted for and this tends to prevent such losses.

3. Many items of farm property, such as machinery and buildings, show a yearly loss of value through depreciation. The amount of the yearly depreciation of all items is shown in each inventory after the first one, and this serves very effectively to emphasize the importance of the proper care of property in order to make the yearly depreciation as small as possible.

There is no doubt that in the long run the farmer will receive a much greater money return for the time spent in taking a yearly inventory than he does for the same amount of time spent in doing any kind of ordinary farm work.

To do practical work on farm inventories in the schoolroom the following steps should be taken: (a) obtain a suitable blank form; (b) obtain information on the normal rates of depreciation of buildings, machinery, and farm animals; (c) arrange for the taking of inventories on some neighboring farms. Use the values of the various items at the time the inventories are taken.

314. Marketing of farm products. — The following example illustrates the application of arithmetic to the problem of marketing farm products to advantage:

Example: One farmer, *A*, sells his corn as soon as it is husked in October receiving 73 cents a bushel, while another farmer, *B*, stores his corn on the farm and sells it the following May at 93 cents a bushel. Did *B* gain or lose by keeping his corn?

Solution: To solve this problem we must know how much the corn shrinks in storage, how much more it cost to handle it twice and to store it than to haul it directly to the market, and how much interest should be charged against the corn while it is lying idle in the crib. Suppose the corn shrinks 16.5% (the average shrinkage of corn from October to May). Suppose further that it cost 2 cents a bushel extra to handle the corn twice and to store it and that the rate of interest is 7%. For simplicity suppose each farmer had 1000 bushels in October. Then *A* sells his corn for 1000 × \$.73 = \$730.

In May, *B* sells 83.5% of 1000 bushels or 835 bushels for 835 × \$.93 = \$776.55. From this must be deducted 2 cents a bushel extra cost on 1000 bushels, or $20, and 7% interest on $730 for 7 months, or $29.81, leaving $726.74. Hence *B* lost $3.26 by keeping his corn.

To solve practical problems of this type it will be necessary to obtain tables showing shrinkages of various farm products from month to month and also to gain some information on the general run of prices at various seasons of the year. In a rural community it should not be difficult to find a boy enterprising enough to get access to the *Yearbooks of the U. S. Department of Agriculture* for ten years past and from them make lists showing how the prices on different products have varied from month to month. From these the average monthly price of each commodity may be obtained, thus providing the class (and the practical farmer) with reliable information as to probable price fluctuations when no exceptional element enters to disturb the normal course of events.

315. Nutritive ratio and balanced rations. — A farmer will not do much reading on the feeding of farm animals before he encounters the expression "nutritive ratio" and such expressions as 1:8.5. We will explain the latter first. The expression, 1:8.5, is an ordinary ratio whose antecedent is 1 and consequent, 8.5. Its peculiarity consists in the fact that the consequent is not an integer. In giving the nutritive ratio of feed stuffs the ratio is always reduced so that the antecedent is unity and this often makes such a ratio have a consequent which is not an integer. (The consequents are usually not carried to more than one place of decimals.) Thus, if the real ratio is 2:11 it will be reduced to 1:5.5 when used to express a nutritive ratio. The reason for this is that when thus expressed the ratios can readily be compared as to magnitude. That is, the two ratios, 5:18 and 7:24, are not easily compared as to magnitudes while the equivalent ratios, 1:3.6, and 1:3.4, are easily compared. Ratios whose antecedents are unity are called *unit ratios*.

In any scientific description of feedstuffs (or of human foods) the reader will find the words "protein," "carbohydrates," and "fats." We can not characterize these completely but we can point out that among human foods lean meat, fish, and certain cereals contain large amounts of protein; sugar contains no food except carbohydrate; while lard is pure fat.

Among feedstuffs for animals, alfalfa hay, peas, and grains in general contain comparatively large amounts of protein, while such feeds as cottonseed meal, linseed meal, and gluten meal are especially rich in this element of food. Carbohydrates are contained to a large extent in all feedstuffs, while fats are less important in animal than in human food. In describing animal foods it is customary to include fats under carbohydrates. In giving the amount of carbohydrates and fats the actual amount of carbohydrates is given and to this is added the amount of fat multiplied by 2.25. The ratio of protein to carbohydrates, as thus computed, is called the *nutritive ratio.*

EXAMPLE: It is known that one lb. of timothy hay contains .028 lb. protein and .456 lb. carbohydrates. Find its nutritive ratio. The solution consists simply in reducing the ratio, .028:.456, to a unit ratio obtaining 1:16.3 which is the required ratio.

PRACTICAL WORK: Obtain a list of the proteins and carbohydrates in various feedstuffs and find their nutritive ratios.

The purpose of the preceding examples is to make the prospective farmer acquainted with the meaning of nutritive ratio and to give him some idea of the range of the nutritive ratios of feedstuffs.

EXAMPLE: A daily ration for wintering cows consists of 35 lb. corn silage, 10 lb. corn stover, and 1 lb. cottonseed meal. Find the nutritive ratio of this ration if it is given that 1 lb. corn silage contains .009 lb. protein and .129 lb. carbohydrates; 1 lb. corn stover contains .017 lb. protein and .370 lb. carbohydrates; and 1 lb. cottonseed meal contains .372 lb. protein and .444 lb. carbohydrates.

Inasmuch as different animals, as well as the same animal under different conditions, require rations of different nutritive ratios, this is an important problem of frequent occurrence. In making a substitution of foods in an approved ration the farmer must take care that the total feed value as well as the nutritive ratio of the new ration is kept within proper bounds.

Practical Work: Obtain a list of the composition of feedstuffs and compute the nutritive ratios of the rations given on page 443 and of rations known to be used in the neighborhood.

316. Statement showing farm income. — Inventories showing the value of all a man's property taken at the beginning and the end of the year would show whether he is richer or poorer at the end of the year than he was at the beginning, and how much. But these inventories would not show how much money he made during the year, for they would not show how much of his income he had spent outside of his business, such as for general living expenses.

If, however, we have a complete inventory of all property used in the farming business both at the beginning and at the end of the year, and if we know how much has been spent in running the farm during the year and also all receipts from it, we shall be able to find how much was made in the farming business during the year.

The following form is used:

(1)	Total inventory at end of year	$47,650.00
(2)	Total receipts (sales) for year	6,595.40
(3)	Sum (1) and (2)	54,245.40
(4)	Total inventory at beginning of year	46,295.00
(5)	Total farm expenses for year	5,152.75
(6)	Sum (4) and (5)	51,447.75
(7)	Net income, (3) less (6)	2,797.65

This is called a *statement*. It shows just how much was made during the year.

Example 1: Make out a statement using the following data: Inventories at beginning and end of year, \$17,845.50 and \$18,264.75; Receipts, \$3,165.47; Expenditures, \$3,274.20.

Example 2: Make out a statement using the following data: Inventories at beginning and end of year \$57,460.40 and \$55,936.50; Receipts, \$14,816.15; Expenses, \$11,242.80.

For *practical work* in making farm statements it would be necessary to obtain not only inventories at the beginning and end of the year but also the total receipts and expenditures. The latter are difficult to get since comparatively few farmers keep track of them completely. This, however, will not detract measurably from the value of this work since the pupil's attention will be called to the meaning and importance of such statements and he will learn to make the statement.

317. Records of farm receipts and farm expenditures. — We have just seen that a farm statement is possible only when total receipts and total expenditures are known. We shall now note more particularly the methods of keeping records of these. The exact form in which such records will be kept depends upon the kind of farm and the information which the farmer desires from his records. Receipts may all be entered in one column or may be divided into a number of columns, each showing the receipts from a particular source. Thus receipts from dairy, poultry, hogs, beef cattle, hay, and grains may appear in different columns. Expenditures may be divided and entered in separate columns in a similar manner. Thus we may have separate columns for expenditures on live stock, machinery, labor, feed, capital expenditures, etc. In the record of both receipts and expenditures there will be one column, showing the total, in which all items are entered. Blank farm account books may readily be obtained which will show how these are entered.

Under "farm receipts" should be entered all receipts that come from sales of farm products and other income that arises directly out of the farm. Thus receipts from sales of grains, hay, farm animals, dairy products, poultry products, etc. should be included.

Under "farm expenses" should be entered all operating expenses, such as feed purchased, wages paid, live stock bought, machinery and tools bought, repairs, taxes, rents paid, farm insurance, interest on farm notes, and all expenditures for farm improvements and for construction of new buildings.

Do not include the value of the farmer's own labor or that of his wife, of unpaid labor of children, or any of the expenditures of the family, such as for food, clothing, or amusements.

For *practical work* it should be possible to get children in the upper grades to become their "fathers' bookkeepers" and in that capacity to keep a complete record of receipts and expenditures, to take the farm inventories, and to make out the statements.

318. Mensuration for the farm. — The principles of mensuration already considered in Chapter XXV will be sufficient to meet the needs of the farmer if intelligently applied. Some of these should, however, receive special emphasis on the part of those who expect to be farmers. Among these are the area of the trapezoid and the triangle. The formula, $a = \sqrt{s(s-a)(s-b)(s-c)}$, may well be taught and used consistently (see section 304). The method of finding the volumes of peculiarly shaped solids so as to know the amount of concrete needed to build a rectangular or a circular water tank may be of real moment to a farmer. The determination of the volume of a concrete post tapering toward the top (not necessarily the frustum of a pyramid), of a flaring corncrib, and of the contents of a tank whose ends are semicircles are real and not infrequent farm problems. Special rules for finding the number of tons in a bin or the number of tons of silage in a silo are important and should be taught.

For *practical work* in farm mensuration a wealth of material will be available in every neighborhood. Haystacks, corncribs, grain bins, etc. may be measured and the desired quantities computed. Fields of various shapes will tax the ingenuity of pupils to get their true areas when no instruments are available except such as are used to measure straight lines.

319. Accounts with farm enterprises. — An account with a farm enterprise will show all elements of expense in connection with it, including the cost of all labor by men and horses, reasonable rent for the land used, the value of seed, etc. On the credit side will be shown all items of receipts, such as for products sold, or the value of products from the enterprise that may have been used on the farm. Thus the credits for a wheat field will show the value of all the wheat harvested, whether it was sold or fed on the farm, the value of the straw, and possibly some credit on account of pasture in case animals were turned into the field after harvest.

In some cases material for *practical work* may be obtained from a farmer who has kept a record of the amount of work required for a certain piece of farm work. It will then be necessary to estimate the value of an hour's or a day's work of man and team. It will always be possible to obtain material for such work by writing to the nearest agricultural college or by referring to a text on farm accounting.

REVIEW QUESTIONS

1. Discuss the need of using arithmetic on the farm. How do the farmer's problems compare with those of the merchant or banker? (§ 310.)

2. How may the farmer save money by computing the cost of animal rations under current prices? (§ 311.)

3. What problems on labor required in farming operations may the farmer solve to his own financial advantage? (§ 312.)

4. Discuss the value of farm inventories and the method of taking them. (§ 313.)

5. Give an example of the kind the farmer needs to solve if he is to market his crops to the best advantage. (§ 314.)

6. Give an example involving nutritive ratios which the farmer needs to solve in order to maintain balanced rations for farm animals. (§ 315.)

7. Describe fully a farm statement showing net profit or loss on a year's farming operations. What records are necessary for such a statement? How may these records be kept? (§§ 316, 317.)

8. What topics in mensuration are important for the farmer? (§ 318.)

9. Describe an account with a farm enterprise. What information of value may a farmer obtain from such an account? (§ 319.)

CHAPTER XXVIII

THE TEXTBOOK, ITS SELECTION AND USE

320. The necessity of a basic text. — In this chapter we are concerned only with texts on grade school arithmetic. The necessity of a basic text is generally recognized though there is a feeling, more or less widespread, that this necessity is due, in part at least, to the inadequate preparation of the teachers. This feeling is not shared by the writer. There are many considerations which would make it inadvisable even for the very best equipped and most talented teacher to attempt to conduct a class in arithmetic without the use of a basic text. The writer has had considerable experience in organizing texts and knows that the labor involved is much greater than any teacher would be justified in undertaking — it would leave no time for anything else. Unless a teacher is willing to spend the years of time in gathering material and organizing it which are represented by practically all standard texts in use, an inferior product would be inevitable. Moreover an attempt to dispense with a regular text entails so much writing on the board, so much dictating, and so much copying on the part of the pupils that much valuable time is lost.

The proper balancing of topics, the providing of drill material at intervals with due regard to the psychology of forgetting, practical applications, careful attention to motivation, all these require prolonged attention. Even the problem of finding local material for applications and for "projects" requires suggestions which can best be given by one who has faced these problems at leisure. Here as elsewhere division of labor is necessary. The practical driver of automobiles may have very definite and constructive ideas as to what the automobile should be, but the incorporation of these ideas into a finished product must be left to those who make the designing of complete cars their special work.

The writer has visited classes taught by the very highest type of teachers who have definite ideas both of the purposes of the course and of means for their attainment. In some of these classes the textbook had been discarded. (In his youthful enthusiasm the writer himself was at times guilty.) The result was invariably a mixture of good and bad. After all, the new ideas being developed by such teachers covered only a minor part of the whole course, and while these parts might be well organized (though this was by no means always the case) the other parts would receive scant attention and almost invariably the course as a whole would be rather inferior.

The really redeeming element in such cases is the infectious enthusiasm of the teacher which partly atones for, but unfortunately serves to conceal, the underlying weaknesses of the work. If the writer may be permitted a bit of advice: If a teacher feels strongly enough that she could improve in an essential degree upon the textbook in use, let her work out a complete manuscript covering the whole subject, not only incorporating her particular ideas but making a serious effort at least to equal the best that has been done in other respects. Then let the finished product be reproduced and placed in the hands of the pupils. Practical use is sure to reveal unsuspected defects which may then be remedied. Only in some such manner can she hope to use her own ideas of organization and her own point of view without subjecting her pupils to the use of material which is sadly inferior in spots. The teacher who tries to incorporate her hobbies in a course must pay the price which has just been indicated. The organization of a subject for instruction after the standards attained in the best textbooks is not an off-hand piece of work.

Not a few texts produced by highly original workers who have contributed much to the understanding of the process of learning are seriously defective in their organization of those elements which are not directly affected by the new ideas which prompted the production of their books.

321. The use of the textbook. — While a basic textbook should be used and while it would seem best that its general plan of development should be adhered to, it by no means follows that there should be a "slavish" dependence on the book. Parts may be omitted provided care is taken that the omitted parts are not needed later in the course. Additional and alternative explanations and applications may and should be given. The adequacy of the amount of drill provided should be considered as the work proceeds and if necessary additional drills provided. It should be kept in mind that the young child (in many cases even the adult) is better equipped to grasp the meaning of the spoken words than of the same words when seen on the printed page. The writer has tested the truth of this statement many times in classes ranging from the elementary grades to advanced graduate work in the university. For this reason new topics should usually be introduced by means of classroom discussion even though the subject may be treated entirely adequately in the textbook. In such cases it is the writer's practice to follow the explanation of the book quite closely so the student may have the opportunity to go over the same matter again when he reads his lesson. Extraneous matter, such as the pointing out of possible practical uses, and especially local applications should of course be introduced. But this first oral development of the topic should be essentially that of the textbook. Other explanations may be offered later when the first one has been mastered.

Unless there are urgent reasons the essential order of the topics given in the textbook should be followed. If the order is inverted, many changes will be required necessitating unexpected labor if the material is to be organized adequately and placed in the pupils' hands in convenient form. If the text in use does not present the various topics in the desired order, the remedy is to exert pressure for a change. Surely some one of the large variety available will be satisfactory in this respect.

In the upper grades increasing effort should be made to have the pupils read the explanations and discussions in the textbook and to reproduce these to the class. While, as remarked above, the spoken word carries the meaning more clearly than the printed page, the school should endeavor to develop the ability to interpret that which is read. If we are to continue seriously the process of learning beyond the period of school life the printed page will be our only instructor. Doctors, lawyers, and engineers are constantly learning by reading, and this is increasingly true of the banker, the business man, and the intelligent farmer. One of the most serious defects of our schools is that people who are educated in them constantly require "lessons" whenever they are to learn anything new.

322. Selecting books with reference to the special needs of the school. — There are several respects in which schools differ sufficiently to raise the question as to whether different types of textbooks would be desirable. Such differences may be inherent in the localities where the schools are situated or they may be due to the different points of view of those who are conducting them. We will consider the latter case first:

Suppose a superintendent and his teachers have come to the conclusion that the work in arithmetic must include a large amount of local material. It is clear that these teachers will not do the best work of which they are capable unless conditions make possible and even highly convenient the carrying out of such a program. A textbook should therefore be selected which stresses this kind of work. It is quite true that the text can not furnish the local material, but it can suggest the points at which it may be brought in and the kind of material that can be used. It is almost impossible to secure local material effectively if this work is deferred until the time it is needed, whereas a list of such material in hand early in the year will make very easy the work of collecting it. Again, the "project method" may be uppermost in the minds of those who conduct the schools. In such a case the text selected should be one which suggests many independent undertakings on the part of the pupils. School people differ on the plan of organization of arithmetic, on the nature of motivation, on the importance of securing a thorough understanding of those practical affairs which are the basis of the applied problems. Some believe that the number concept is based most naturally on accounting, and others that it has its natural basis in measuring. Some regard the course in arithmetic purely as a practical matter valuable only for its information and habituated processes; others believe that it develops a critical attitude and an understanding of the possibility of overcoming difficulties which at first seem too complicated to be mastered. Some believe that each problem should be solved "from the ground up" by a thoroughgoing analysis of that problem; others think it possible and profitable to develop general ideas which will serve in the solution of hosts of problems. Some believe it of fundamental importance that the principles of arithmetic be raised to the conceptual level and as such kept constantly in the foreground; others lay no great stress on this phase of the subject. Unless we are overzealous in our missionary spirit to convert others to our point of view, the natural advice would be to permit each group of teachers to use a text which would be the most effective in doing the work in the manner in which they think it should be done.

It is sometimes urged that a school located in an industrial section of a city should use a different textbook from one where the children are mostly from the homes of business and professional people. This point may be open to doubt. Let us consider a class of thirty pupils in a school in an industrial section. There are say fifteen boys and fifteen girls. Approximately twelve of these girls will be housekeepers. No reasonable prediction can be made as to the occupation of the other three. They may become clerks (saleswomen), stenographers, teachers, nurses, houseworkers, private secretaries, librarians, etc. We can not predict with any degree of certainty that a single one of the fifteen boys will enter a specified occupation. In an industrial district in Detroit, for example, it may be safe to assume that some will go into the automobile industry, but this industry has literally hundreds of essentially different occupations. A glance at a book listing different occupations and describing the qualifications required for each will be highly instructive on this point.

The future occupations of the members of a class where the pupils are mostly from the homes of business and industrial people are equally uncertain, again with the exception that a large percentage of the girls will be housekeepers.

From these considerations it would seem highly doubtful whether essentially different subject matter should be included in the courses in arithmetic given in these two classes. It is of course true that the two groups are equipped with somewhat different experiences which must form the background of the course, but these vary so constantly from place to place that they can not be taken account of in the text. All that can be expected of the text is that it makes use of material which is more or less within the range of experience of all wide-awake children. The more minute adaptations must be secured by bringing in local material. The only thing that can be said with certainty is that stressing arithmetic for the home will prepare directly for the work of about forty per cent of the members of a mixed class.

We must not deceive ourselves by observing the conditions in night schools whose classes are largely made up of those who have already entered into industrial life and who therefore have very specific practical needs. It would be wholly indefensible to assume that the very special applications of arithmetic which are demanded in such classes are at all adapted to the needs of general classes in the grammar school.

It remains to consider the rural schools. Here a large proportion of the boys will be farmers and it is important that the special problems of the farm should be emphasized (see Chapter XXVI). Indeed it is of considerable importance that the girls even should study such problems. Not infrequently the intelligent wife of a farmer is instrumental in keeping important farm records. If she does not keep the records herself, it is nevertheless important that she understand their purpose and meaning. Appreciative understanding is no small element in motivating the less obviously necessary work of both husband and wife (see section 305). Moreover, the environments in the city and the country are very different—the distinction being much more pronounced than between different rural districts or between different cities. Hence the child in the rural districts has a very different experiential background from the child in the city. It would seem therefore that there should be distinct textbooks in arithmetic for the rural schools — not merely rural arithmetics whose contents may be superimposed on the regular course but a distinct series for all grades having their "center of gravity" in rural life and entirely adequate for the whole course in arithmetic.

323. The textbook as a teacher's aid. — The textbook should be prepared not only with a view to making it the best possible instrument for learning on the part of the child but also with a view to making it the best possible instrument of teaching in the hands of the teacher. Perhaps this latter aim can be achieved most effectively through a separate teachers' edition. There are real objections to "suggestions to teachers" in the main text, and these objections are valid, in part at least, against notes for teachers placed at the end of the book which is in the hands of the pupils. The preface to the notes in the special teachers' edition of a series of arithmetics of which the writer is one of the editors runs as follows:

> One purpose of the teachers' edition of these books is to provide detailed suggestions to the teacher. Many inexperienced teachers have had little or no normal training in the art of teaching, and even those who have had such training frequently find it difficult to make practical application of general principles to each day's work. For all such teachers these notes should be of decided value.
>
> A second purpose is to provide in convenient form much of the material necessary for the best kind of teaching, thus setting the teacher free from a large share of the drudgery that has too often been regarded as inseparable from her work.
>
> This material includes:
>
> (*a*) A list of local material for supplementary work.
>
> (*b*) Material and suggestion for drills in the fundamentals at frequent intervals throughout the book. Answers are given to all proposed exercises, thus relieving the teacher of unnecessary work in performing the computations.
>
> (*c*) A series of standardized tests in the fundamentals which will enable the teacher to decide whether or not her classes are up to the normal standards in the four fundamental operations.
>
> (*d*) Six sets of material for examinations are provided for each year's work. This will relieve the teacher of the trouble of finding suitable material for examinations and the labor of performing the computations.
>
> (*e*) All problems involving two or more steps are solved, showing the result of each step. This has been done not because there may be some exceptionally weak teachers who do not know how to solve the problems, but because it is desired to relieve *all* teachers of the work of performing the computations. In checking papers the teacher needs to know in what step an error may have been made. Hence the result of each step is given.
>
> (*f*) The answers to all problems and exercises are given in the text in distinctive type and on the same page with the problem and exercises them-

selves. This saves the teacher the trouble of looking in the answer book to verify the answers.

(*g*) Except for the answers just mentioned, the page of the main text in the teacher's edition is exactly like that of the pupil's book. It is therefore unnecessary for the teacher to ever use the pupil's book.

It has been customary to regard a teacher as lazy or indifferent if she seeks to avoid doing any of the work that she asks her pupils to do. Nothing could be more unfair and misleading. There are good reasons why young children should spend much time on drills in the fundamentals, but there is no reason why a competent teacher should spend any time whatever on such drills. Teachers, supervising officers, and makers of textbooks should constantly bear in mind that whenever the teacher is relieved of unnecessary drudgery she is thereby set free for the more essential parts of her extremely difficult work of leading onward in the best way the young minds entrusted to her care. To this end the authors of these books have worked with singleness of purpose.[1]

324. Free textbooks. — It is not the purpose here to give even a résumé of the discussion that has taken place about the relative advantages and disadvantages of free textbooks. The arguments have been nicely balanced—about the same number of points on each side.[2] The relative weight that one assigns to these arguments *pro* and *con* will depend largely upon one's general views as to the proper functions of government. One of these points has already been noted under the caption, "The pupils' text as a permanent reference book" (see section 176). This point seems to the writer the most fundamental in the whole discussion and curiously enough it is the one which has received the least attention. It is usually not stated explicitly though it may be implied in the statement that when the pupils buy their own books they will be making a start in accumulating a library. That, however, scarcely tells the story. We are concerned with a very unostentatious part of a library—a dictionary of the English language, outlines of history stating the salient facts and giving their chronological order, maps and concise descriptions of parts of the earth, business usages, tables and processes involved in practical computation—the naked skeletons which will be clothed by auxiliary reading in school and general reading later. These books contain certain fundamental facts, of which we are in more or less frequent need throughout life. In school we become intimately acquainted with them, and it would not even seem open to argument that they should be retained and placed where they are most easily accessible. The full importance of having books where they may be referred to most easily is frequently not appreciated. The writer has had his dictionary in various positions with reference to his desk. Once it was placed at a distance so it was necessary to walk a few steps to reach it. Then it was moved so it could be used by turning about in a revolving chair; now it lies on the desk itself. Each of these changes has increased very largely the extent to which it is used. If it were located in the next room or in the city library, it would be used very seldom indeed.

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[1] Lennes and Jenkins, *op. cit.* (p. 150), Vol. II, pp. 295, 296.

[2] C. R. Maxwell, *The Selection of Textbooks;* Houghton Mifflin Company, 1921, pp. 45–46. Alfred Lawrence Hall-Quest, *The Textbook;* The Macmillan Company, 1918, pp. 51–54.

The writer suggests the following compromise as the best way of meeting the various difficulties. A list of books could be made, consisting of all those to which there is likely to be reference in later school work or in practical life, and these should be the exclusive personal property of the pupils. The community spirit is fine, but the practical carpenter must have his own tools in a convenient place so he can pick up such as he needs without loss of time or diversion of attention. For the present purpose it would not matter whether these books were bought by the child's parents or by the school district, but they should be his own and we should use all the means at our disposal to induce him to keep them. Other books, such as elementary readers, language books, spellers, and supplementary books of all kinds, might be furnished free and the property right in them retained by the school. Even here questions might arise. Should not children be encouraged to acquire for their permanent possessions such beautiful books as some of our geographical readers, animal stories, or biographies?

It is not unlikely that in this controversy as in so many others those who arrive at opposite conclusions are partly led to do so by considering in the main different parts of a complicated situation. Whichever way such a question is decided there will be real disadvantages as well as advantages. May it not be possible that the "mixed" solution suggested above will meet more completely the views of those who have considered this matter than all free books or no free books?

325. The selection of textbooks. — There is no doubt that the work of selecting textbooks can be facilitated and be made more effective by the adoption of a definite method of comparing the books that are under consideration. When this is done the usual method is to agree upon a series of qualities and the weight to be given to each and then to grade the books on each of these qualities. Maxwell[1] gives the following analysis of a text on arithmetic, the book being graded as excellent, good, fair, and poor for each quality:

- I. Point of view:
    - 1. Mental training
    - 2. Social efficiency
- II. Organization of subject matter:
    - 1. Topics developed logically and psychologically:
        - A. Development topical, spiral, or cumulative
    - 2. Nature of the problem material:
        - A. Amount
        - B. Practical
        - C. Conform to business practice
        - D. Within experience of children at this stage of development
        - E. Properly graded
        - F. Provision for individual differences

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[1] *Op. cit.* (p. 461), pp. 92–96.

        G. Economy of time
        H. Language clear, concise, nontechnical, easily understood, corresponding to vocabulary of readers.
        I. Proper attention to thrift and other important economic principles best taught in arithmetic.
    3. Drill material:
        A. Adequate
        B. Practical
        C. Prepared with results of standard tests in mind
        D. Variety
        E. Provision for motivation
        F. Provision for individual and group differences
    4. Attention to business forms:
        A. Simple accounts
        B. Experience with checks drafts, receipts and other common business forms
    5. Preparation for other mathematics:
        A. Use of equation
        B. Purpose of equation
        C. Use and purpose of graphs
        D. Construction
    6. Elimination:
        A. Basis
        B. Topics
        C. In accordance with recommendations of committee on economy of time
III. Methods of presentation:
    1. Inductive — principles involved
    2. Deductive — principles given
    3. Suggestions for use of local data
    4. Use of objective material
    5. Suggestions for teaching topics
    6. Other teaching helps
    7. Answer book for teachers, separate pamphlet for students, answers bound in book
    8. Influence of standard tests apparent
    9. Emphasis upon development of fundamental habits of estimating answers, checking results, etc
    10. Training in independence of paper and pencil — mental work

Hall-Quest[1] gives several sets of standards that have been proposed for arithmetic texts. From these we select a set formulated by a committee on textbook recommendations in Cincinnati:

1. Textbooks in arithmetic should be judged with reference to the provisions which they make for relating the child to the more important number aspects of his life in society.

2. The *problem form of organization, i.e.* each new phase in arithmetic should be approached and developed as a problem growing out of the interests, experiences, and needs of the child. How can the mastery of this process, formal or concrete, help me, as a pupil, to meet and solve a situation I am interested in?
   - *a.* The following sorts of problems: those of the home, playground, school, social, and industrial life.
   - *b.* There should be much material from which the child has to make a choice from among data not all of which are involved in the solution of the problem.
   - *c.* Many problems may at times not involve numbers but may present situations which demand explanation.

3. The following material and processes are considered *obsolete* and are largely *eliminated* by most texts and courses of study of the last ten years:
   - *a.* The horizontal placing of numbers in the various processes
   - *b.* The old form of placing the quotient in division, with the old continued method of pointing off in division of decimals
   - *c.* Greatest common divisor and least common multiple
   - *d.* Unreal, simple, and complex fractions
   - *e.* Reduction in denominate numbers of more than three places
   - *f.* Rules and explanations for processes not to be explained in light of modern authority, as inverting and multiplying in the division of fractions
   - *g.* Formulas and rules of methods in percentage
   - *h.* Troy and apothecaries' weight
   - *i.* Longitude and time with more than 15 degrees unit
   - *j.* True discount
   - *k.* Partial payments
   - *l.* Annual and compound interest

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[1] *Op. cit.* (p. 461), pp. 95–98.

m. Foreign exchange
n. Equation of payments
o. Cube root
p. Topical plan of arrangement

4. *Children's interests* and activities should be provided for in the way of number games, construction work, etc. Work in the fundamentals should grow out of such material.

5. *Pictorial aids.* Illustrative pictures; diagrams, tables, graphs; devices should be numerous.

6. *Excellence of drill.* Frequency. A large factor is the amount of *motivated* drill work.

7. Provision for *individual differences* of pupils. There is a growing need for this. There should be a minimum for all; but provision should also be made for additional work for brighter pupils.

The following score card is proposed by this committee:

1.	The problem form of development	100
2.	Skillful motivation of drill work and reviews	150
3.	The nature of the thought problems	250
4.	Provision for enlargement of number concept	25
5.	The exclusion of obsolete material and worth of new material	100
6.	Regard for relative value of topics included	
7.	Adaptability to children of the average community	75
8.	Easy grading of processes and problems	100
9.	Suggestions for work of local interest	25
10.	Appendix and Index	50
11.	Respect for judgment of the teacher	25
12.	Relevancy of illustrations	25
13.	Mechanical make-up	25

REVIEW QUESTIONS

1. Discuss the necessity of using a basic textbook even when the teacher is adequately prepared to organize the course according to her own plans. (§ 320.)

2. Discuss the use of the textbook, considering the following questions: (*a*) Should the order of the book be followed in class? (*b*) Should the type of explanations given in the text be followed in classroom explanations? (§ 321.)

3. Discuss the comparative difficulty of understanding oral and printed explanations. Discuss the importance of "learning to read." (§ 321.)

4. Consider the question as to whether or not teachers with different aims in their teaching should use different texts. If this question is answered in the affirmative, what bearing has the answer on the policy of selecting the same book for a large area (such as a large state) including schools of widely different types? (§ 322.)

5. Discuss the question as to whether different types of books should be used in different sections of a city on account of the different industrial groups from which children come. (§ 322.)

6. Consider the textbook as a teacher's aid. In what ways may the text relieve the teacher of drudgery? (§ 323.)

7. State the argument against free textbooks given in section 324. What substitute for the free textbook plan is proposed in this section?

8. Discuss various methods of selecting textbooks. (§ 325.)

CHAPTER XXIX

MEASURING THE RESULTS OF TEACHING

326. Examinations set by the teacher. — Every teacher seeks to ascertain by various means the progress of her pupils. Items of information on this point obtained from formal examinations, from oral questions during recitations, and by taking general note of the character of all classroom performances are given varying relative weight in arriving at the conclusion which she finally embodies in definite grades. The grades given by different teachers for a given performance must of necessity vary with the point of view of the teacher and with the general spirit of the school system in which she is working. In an examination what relative weight shall be given to problems closely similar to those studied in the class and which therefore require little originality on the part of the pupil who has done his work faithfully and other problems not so closely related to the types that have been studied and whose solution therefore requires much more real thinking? Again, one teacher fixes her grades primarily with reference to the relative excellence of the members of her class, the absolute grades being so adjusted as to conform to the standard probability of success and failure, while another teacher has a more or less definite standard of absolute achievement in mind. All experienced teachers know that whatever may be the case when very large groups are considered, small groups such as constitute a single class differ very considerably among themselves in average ability. The writer has taught literally scores of classes in plane trigonometry both in college and in the high school, and he has no hesitation in affirming that the greatest average ability found in these was in certain classes in a large high school where this work was taken only by those who were particularly effective in mathematics. How shall such a class be graded? The answer must depend more or less upon the attitude of the teacher.

Similar considerations apply even to the fundamental operations in arithmetic. In one school or in one class particular stress is put upon accuracy. The objective is absolute accuracy in all work handed in for inspection. Such accuracy can never be attained except by the use of checks which require considerable time. Suppose ten examples in multiplication are given. A paper containing eight correct results and two that are wrong may be marked 80 by one teacher and very low by another, while the latter teacher would mark six examples properly checked very high.

In such a complex subject as arithmetic a large number of abilities and interests are brought into play and the possible combinations of these produce an endless variety of net results. Of two pupils of very different capacities, one is of the dutiful kind who has practiced the fundamental operations faithfully and thus achieved a very respectable standard of speed and accuracy while problems requiring reasoning cause no end of trouble; the other can reason out any problem that is set with very little difficulty and with practically unerring accuracy and has very clear ideas as to the functions of arithmetic in practical life, while his interest in mechanical drills has remained so low that his speed and accuracy in the fundamental operations leave much to be desired. How are these pupils to be graded? Again the answer must depend upon the point of view of the one finally assigning the grade.

Finally, grading a paper is to a considerable extent a matter of judgment and of personal attitude even when general principles are laid down and followed. Shall an answer which admittedly has some defects be graded 6, 7, or 8 on a scale of 10? The cynical might say that this would depend much upon the weather or the personal condition of the teacher and probably the cynic would not be far wrong; and a difference of only one point for each question in a set of ten might spell success or failure.

327. Value of grades given in the schools. — The considerations of the preceding section lead inevitably to the conclusion that no high degree of agreement may be expected when a certain school performance is graded by different teachers. That the actual relative difference in the grades of two pupils who are graded by several teachers in their progress through the grades is not greater than it is is no doubt due to the adaptability of the pupils. One teacher in her class work lays particular stress on certain phases of her subject and these are naturally emphasized in her examinations and in her final judgment on which the grades are based. The "getting used to a teacher" lies in no small degree in adjustment to her particular emphasis. Our knowledge of this difference among teachers has been made precise by certain investigations:

(*a*) *It has been found that the same pupils are given different relative and absolute grades in different schools.*

Monroe [1] summarizes the results obtained by Kelly [2] as follows:

> In 1913, Kelly made an investigation of the marks given to the sixth-grade pupils in four ward schools in Hackensack, New Jersey, and the marks given to the same pupils when they went to a common departmental school for seventh-grade work. This will be recognized as a case where the abilities of the same pupils were measured by two different sets of teachers, the sixth-grade teachers in the ward schools and the seventh-grade teachers in the departmental school. Since in the departmental school all of the pupils were taught arithmetic by one teacher, there was an opportunity to compare the "grades" given in arithmetic by the sixth-grade teachers in the different ward schools. If these teachers were accurate in their "grading," we would expect to find that all of the pupils who received a mark of "G" (good) in arithmetic in the sixth grade would receive approximately the

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[1] Walter Scott Monroe, *Measuring the Results of Teaching;* Houghton Mifflin Company, 1918, pp. 3, 4.

[2] F. J. Kelly, *Teachers' Marks*, Teachers College Contributions No. 66, 1913, p. 7.

same mark in the seventh grade. If, however, the sixth-grade teachers were inaccurate in their marking, — that is, some of them marked too high or too low, we would expect to find that pupils having the mark of "G" in the sixth grade, but coming from different schools, would, on the average, receive different marks in the seventh grade. This condition was found to exist. Kelly states his conclusions as follows: This means that for work which the teacher in school "C" (one of the ward schools) would give a mark of "G" (good) in language, penmanship, or history, the teacher in school "D" (another ward school) would give less than a mark "F" (fair).

(*b*) *Different teachers give very different grades to the same paper.* Starch and Elliott studied the accuracy with which teachers mark papers in English, geometry, and history [1] and Monroe reports the investigation as follows:[2]

> A facsimile reproduction was made of an actual examination paper in plane geometry. A copy of this reproduction was sent to each of the high schools included in the North Central Association of Colleges and Secondary Schools, with the request that it be marked on the scale of one hundred per cent by the teacher of geometry. The teacher was asked to mark the paper by the method he was accustomed to use. Papers were returned from 115 schools, and the results tabulated. When we consider that the subject matter of geometry is quite definite, and that the papers were marked by teachers who were thoroughly acquainted with the subject, it would seem that we might expect the marks or "grades" placed upon this examination paper to be in close agreement. However, exactly the opposite was the case.

The results were as follows: below 55, 13 grades; 55 to 59, 7 grades; 60 to 64, 17 grades; 65 to 69, 18 grades; 70 to 74, 13 grades; 75 to 79, 27 grades; 80 to 84, 11 grades; 85 to 89, 7 grades; 90 to 100, 2 grades. This shows an average deviation of 7.5 points from the medium grade — a greater deviation than was found in the grading of papers in English and history which were 4.0 and 4.8 respectively. On this point the remark by Starch [3] is illuminating:

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[1] Starch and Elliott, *School Review*, Vol. 20, pp. 442–57; Vol. 21, pp. 254–59 and pp. 676–81.

[2] Monroe, *op. cit.* (p. 470), p. 8.

[3] Daniel Starch, *Educational Measurements*, The Macmillan Company, 1917, p. 7.

> While it is true that there can be no difference of opinion as to the correctness of a demonstration or of the solution of a problem, yet there are various ways in which a demonstration may be worked out, involving the succession of the steps, the use of theorems and definitions, the neatness of the drawings, and, most of all, the relative value of each particular demonstration, definition, step or error in the evaluation of the paper as a whole. Obviously the complication of factors is as intricate in one sort of paper as in another.
>
> It is therefore evident that there is no inherent reason why a mathematical paper should be capable of more precise evaluation than any other kind of paper. In fact, the greater certainty of correctness or incorrectness of a mathematical demonstration or definition may even contribute slightly to the wider variability of the marks, because the strict marker would have less occasion to give the pupil the benefit of the doubt.

(*c*) *The same teacher gives different grades to the same paper when grading it at different times.*

Starch[1] makes the following statement:

> By having seven instructors regrade each a set of his own papers, it was found that the mean variation was reduced to 2.2 points. In several instances the second mark differed as much as ten or fifteen points from the first mark.
>
> The mathematics instructors did not agree any more closely with their own marks than the language or science instructors. We may eliminate one further factor; namely, the difference due to a change in an instructor's standard after an interval of time. This was done by weighting the second set of marks by the difference between the averages of the two markings. This reduced the mean variation to 1.75 points.

Starch[2] enumerates the following factors as contributing to the differences in the grades given to the same examination papers:

> Of these factors, each contributes the following amount to the total variation: The general mean variation or probable error of grades assigned by teachers in different schools was 5.4 points. The mean variation of grades assigned by teachers in the same department and institution was 5.3. The mean variation of the latter, after eliminating the effect of high or low

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[1] *Op. cit.* (p. 471), p. 9.

[2] *Idem.*, p. 8.

personal standards was 4.3. The mean variation of grades assigned at different times by the same teachers to their own papers was 2.2. Hence the largest factors were the second, third and fourth. The fourth contributed 2.2 points, the third 2.1 points, the second 1.0 point and the first practically nothing toward the total of 5.4 points of mean variation.

328. Standard tests in the four fundamentals. — With the avowed purpose of remedying some of the defects in the ordinary method of grading, standard tests have been devised. It is clear that if the same tests, administered under closely similar conditions and graded according to the same standards, are used in different schools, the comparative abilities of the pupils can be measured quite closely in so far as these abilities are covered by these tests. In the case of the four fundamental operations absolute standards of achievement have been determined. Such standards require that a pupil in a certain grade should be able, for instance, to finish correctly in say six minutes a certain number of examples, of a certain degree of difficulty, in addition. This standard of speed has been determined by taking the average speed of a large number of pupils in each grade, the standard being the level reached or exceeded by one half the pupils tested. Courtis [1] the author of a widely used set of tests in arithmetic says:

> The speeds (rates) set as standard are approximately the average speeds (rates) at which the children of the different grades have been found to work when tested at the end of the year, when for any one grade a random selection of five thousand scores from children in schools of all types and kinds are used as a basis of judgment.

Accompanying each set of tests are detailed directions describing the conditions under which the tests are to be given. It is important that these directions be followed closely since it has been found that "slight changes in the method of giving the tests may affect the scores as much as the difference in the standard from one grade to another."[1]

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[1] S. A. Courtis, *Third, Fourth, and Fifth Annual Accounting;* Department of Coöperative Research, Detroit, 1913–16, p. 49.

Efforts have also been made to develop standards to be achieved in solving problems involving reasoning. The method for determining the standards is the same as that used for the fundamental operations; namely, the speed and accuracy with which large numbers of pupils in each grade actually solve the problems.

We will now describe certain standard tests more in detail.

$$\begin{array}{r} 927 \\ 379 \\ 756 \\ 837 \\ 924 \\ 110 \\ 854 \\ 965 \\ \underline{344} \end{array}$$

The Courtis Tests,[2] Series B, consist of four tests, one on each of the fundamental operations. Each test contains twenty-four examples which are closely equal in difficulty. Thus all of the examples in the addition tests contain nine three-figure numbers. (The first of these is given in the right margin.)

Each example in the subtraction test, one of which is shown here, contains eight subtractions. The examples in multiplication and division are of the types $\begin{array}{r} 8246 \\ \underline{92} \end{array}$ and $94\overline{)85352}$. The time given for the addition test is 8 minutes; for the subtraction test, 4 minutes; for multiplication, 6 minutes; and for division, 8 minutes.

$$\begin{array}{r} 75088824 \\ \underline{57406394} \end{array}$$

When the tests are given the pupils are given the following directions:

> You will be given 8 minutes to find the answers to as many of these addition examples as possible. Write the answers on this paper directly underneath the examples. You are not expected to be able to do them all. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

The following standard scores at the close of each school year are based on the performances of about 25,000 pupils:

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[1] Monroe, *op. cit.* (p. 470), p. 108.

[2] Published by the World Book Company.

GRADE	ADDITION		SUBTRACTION		MULTIPLICATION		DIVISION	
	Attempts	*Rights*	*Attempts*	*Rights*	*Attempts*	*Rights*	*Attempts*	*Rights*
3	4.0	2.0	4.0	1.0				
4	6.0	3.0	6.0	3.0	4.5	1.5	3.5	1.0
5	7.5	4.0	8.0	5.5	7.0	4.0	5.0	3.0
6	9.0	5.0	10.0	7.0	8.5	5.5	6.5	5.0
7	10.5	6.5	11.5	8.5	10.0	6.5	8.5	7.0
8	12.0	8.0	12.5	10.0	11.5	8.0	10.5	9.0

Each pupil is given two grades, one for the number attempted and one for the number of correct answers.

It would seem to the writer that as a final record of achievement and as an objective to be worked for only the number of correct answers should be considered. Rapid but inaccurate work is of less practical value than much slower work with high degree of accuracy. Indeed one might make a good argument for giving a higher grade to a paper with 12 attempted and 10 rights than to one with 15 attempts and 10 rights. For the purpose of teaching it is valuable to know, however, that a pupil is, say, relatively inaccurate, relatively slow, etc.

329. Standard reasoning tests. — Tests designed to measure the reasoning ability in children have been proposed by Stone,[1] Courtis,[2] Bonser,[3] Starch,[4] Monroe,[5] and others. Naturally the difficulty of standardizing such tests is much greater than in the fundamental operations. There is not even general agreement as to method. Some (*e.g.* Courtis) set a definite time limit within which the reasoning must be done, while others (*e.g.* Starch) permit the child to work on a series of problems of increasing difficulty until he can go no further.

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[1] C. W. Stone, *Arithmetical Abilities and Some Factors Determining Them;* Teachers College, 1908, pp. 11, 12.

[2] *Op. cit.* p. 474.

[3] F. C. Bonser, *The Reasoning Ability of Children,* Teachers College, 1910, p. 2.

[4] *Op. cit.* (p. 471), pp. 114–116.

[5] *Op. cit.* (p. 471), p. 155.

330. Criticism of tests. — Standardized tests in the fundamentals certainly serve as a fairly accurate measure of ability in these. The elements of unreliability of examination grades enumerated in section 327 are almost entirely absent. The greatest defect of some of the most popular of these tests is probably that the same tests are used for all grades. The examples are no doubt too hard for the fourth grade and too easy for the eighth.

One of the important facts revealed by these tests is the large percentage of inaccuracy. Courtis says:

> At present in addition and multiplication it is only very exceptional work in which the median rises above eighty per cent accuracy, while in subtraction and division the limiting level is ninety per cent.

An interesting point is raised as to whether or not children should be drilled to exceed the so-called "standard" speeds. To decide this question regard must be had to the way the standards have been fixed. (See quotation from Courtis given below.)

In many schools from which records were taken, surely no serious efforts were made to secure reasonable speed. Hence the standard scores should be very easy to equal or exceed; with a reasonable amount of properly motivated drill such as is suggested in the earlier chapters of this book, they will be exceeded by a considerable margin.

An opinion of quite general vogue is expressed as follows by Courtis:

> Standard scores are not only goals to be reached; they are limits not to be exceeded. It seems as foolish to overtrain a child as it is to undertrain him. All direct drill work should, in the judgment of the writer, be discontinued once the individual has reached standard levels.

Doubtful as is the propriety of adopting the standards set as aims to be equalled but not exceeded by the median of a class, it is very much more doubtful when applied to each individual pupil as seems to be advocated in the statement just quoted.

Many pupils can easily double the "standard" rate and there would seem to be no reason why they should not be made to do so. Giving an undue amount of time to mechanical drills should be avoided in any case; it may be no more unreasonable to endeavor to bring a very slow child up to the "standard" than to bring another to a speed twice as great.

The tests in reasoning are of much more doubtful value. Indeed results obtained from them would seem to suffer from all the defects of examination grades enumerated in section 327. In some respects they are inferior to the latter. The teacher knows just what types of problems have been solved by the class and the elements that have been stressed, and her examination is set with this in mind. In the case of a test prepared by someone other than the teacher, a large element of chance enters. By good luck a class may be specially prepared for the test or by bad luck it may be very badly prepared for it. The instances given on page 469 show some of the elements that may contribute to the success or failure of a class in such a test. Anyone who has had experience in "coaching" students for examinations knows of other important elements that help to determine whether such tests will be passed successfully or not. In case much depends upon high scores in such a test, there are very good reasons why a test given the publicity necessary to standardize it should not be used. One important item in the equipment of a successful coach for an examination is the possession of questions set in this examination for years past. The inference is obvious.

331. Diagnostic tests. — All tests except the "final examination" are diagnostic in the sense that they furnish information about the weakness both of the class as a whole and of the individual pupils. Obviously there are many special types of difficulties which a child encounters in learning arithmetic. Any well-organized text presents such difficulties separately to be mastered one at a time. Later it will be found that a particular child really did master some of these and possibly not others. In addition of integers, for example, there are first of all the 45 fundamental combinations each of which represents a separate and special difficulty. Then comes the adding within a decade such as 32 + 5, the "bridging of the tens" such as 47 + 8, and finally carrying. There is also the difficulty of keeping the attention closely enough on the work to add a column of considerable length. From time to time the discerning teacher will give diagnostic tests to discover the particular elements of weakness in her pupils. This will enable her to give additional drill where most needed and will show each pupil just what he needs to do to bring up his work.

REVIEW QUESTIONS

1. Describe the various elements of uncertainty or error that may enter into the marking of examination papers. (§ 326.)

2. Show how the teacher's personal point of view must influence the grades she gives. In what way does this fact influence comparative grades which different teachers give for the same work? (§ 326.)

3. State the result of the investigation of F. J. Kelly, described on page 470.

4. State the results of the investigation of Starch and Elliott, described on page 471.

5. Discuss standard tests, (*a*) in the four fundamentals; (*b*) in reasoning. To what extent are standard tests in the fundamentals free from the uncertainties of tests given by the individual teacher as noted in section 326? To what extent are standard reasoning tests free from these uncertainties? (§§ 329, 330.)

6. What are some adverse criticisms of our present standard tests? (§ 330.)

7. What are diagnostic tests? Discuss their purpose and use. (§ 331.)

INDEX

Abilities, correlation of, 31; doctrine of nontransfer of, 44; hierarchy of, 16; synthesizing of, 16; transfer of common elements in, 16; variations in, to reach a desired end, 39.

Ability, analysis of, 15; common factors in, 17; composite character of, 15; decrease in, due to fatigue, 36; defined, 13, 14; determined by experimental psychologists, 7; developed by training, 19; equation of, 15; gaining of, to make money, 46; intellectual, analogy with physical strength, 4; of reasoning, 31; spread of, 8; to overcome fatigue, 37–39.

Abstract numbers, rules for operations on, 314–318.

Accuracy, of language in arithmetic, 181–183; of product of decimals, 274–276; standard of, in arithmetic, 175, 176.

Activity, pleasure of, as a motivating force, 131, 132; dependent upon one's present adaptation for it, 152, 153.

Addends, 315; interchange of, 205–207.

Addition, by endings, 221–224; carrying in, 223–226; checking results in, 226; column, 224; comparative difficulty of, 197–202; interchange of addends in, 205–207; memorizing necessary in, 197; of decimals, 270, 274; of fractions, 249, 261; relations, 177; solution of problems in, 297, 298, 300–302.

Addition-borrowing method of subtraction, 231–234.

Addition-carrying method of subtraction, 231–234.

Adults, imitation of, 157; learning of, 77–80; motives which actuate, 160–162.

Aims of education. *See* Education.

Ambition, as a motivating force, 163–165.

Amount, defined, 303. *See* Interest.

Analysis, solving one-step problems by, 318–321.

Angell, James R., 16, 21, 22, 165, 166.

Angell and Moore, 9.

Areas, computed by mensuration, 428–432.

Arithmetic, accuracy of language in, 181–183; addition, 197–202; aim in, 175, 176; and allied topics, 172–180; child's present need of, 138–140; commission in, 341–343; commutative law of addends, 205–207; defined, 174; development of critical attitude in, 188, 189; discount, 329–335; division, 211, 212, 238; drills on fundamentals in, 245; for the farm, 441–450; for the home, 433–440; fractions, 211, 212, 249–277; graphs, 412–419; in grammar grades, 172–174; insurance, 403–410; interest, 344–355; measurement and denominate numbers, 278–287; memory work, necessary in, 175, 178, 195, 197; mensuration, 426–432; motivated by interest in human affairs, 157, 158; multiplication, 197, 202–204, 234–238; multiplication table, 211–213; necessary qualities of a good course in, 171, 172; objects to be achieved in, 171–190; operations on whole numbers in, 221–248; percentage, 289–293; practical needs of fundamental processes in, 246, 247; principles *versus* memorized rules in, 183–186; profit and loss, 336–340; rationalizing the processes in, 243–245; savings and investments, 357–370; solution of problems in, 294–327; square and cube root, 420–425; stocks and bonds, 371–382; subtraction, 207–210, 227–234; taxes, 395–401; usefulness of, not a motivating force, 119.

Austrian method of subtraction, 209, 210, 232–234, 304.

Bagley, W. C., 24.
Bair, 9.
Bank discount, 389.
Bankers', drafts, 393; method of computing interest, 349–351.
Banks, 384–390; commercial, 386; loans made by, 387, 388; method of studying, 389, 390; practical importance of the savings, 385, 386; rates of interest paid by savings, 384, 385; savings, 384; service of, in paying bills at a distance, 393, 394; study of checking accounts in, 386, 387.
Bennett, 10
Bergstrom, 9.
Berman, Louis, 85.
Bonds. *See* Stocks and bonds.
Book, 10.
Brown, J. C., 31, 41. *See* Brown and Coffman.
Brown and Coffman, 176, 177, 183, 184, 191, 245, 246, 267, 272, 273, 281, 286, 289, 318, 323, 324.

Cairnes. *See* Jastrow and Cairnes.
Carrying, in addition combinations, 223–226.
Charters, W. W., 164, 176.
Checking accounts, study of, 386, 387.
Coffman, L. D., 31, 41. *See* Brown and Coffman.
Combinations, child's list of difficult number, 204, 205; column addition of fundamental, 221–224; drill on fundamental, 215–218; listing, already learned, 212, 213; mastering, separately, 213, 214; order of introducing number, 214, 215.
Commission, 341–343; basis for computing, 342, 343; defined, 341; kinds of business done on, 341, 342; occupations requiring a knowledge of, 343.
Common elements, fallibility of solutions based solely on, 55–57; in practice work and life work, 45; in processes of learning, 58, 59; psychologists' limits of, 20, 21; transfer of, in abilities, 20.
"Common factor," analysis of, 17.
Common fractions. *See* Fractions.
Commutative law of addends, 205–207.
Concept, number, 191–193.
Concrete numbers, rules for operations on, 314–318.
Confidence, produced by success, 53–55.
Coover, J. E., 7, 11, 17, 18, 19.
Coover and Angell, 10.
Counting, and measurement, 192, 278, 279; basis for, 191, 195; process of, 191, 192.
Cowling. *See* Judd and Cowling.
Critical attitude, development of, 188, 189.
Criticism, modern, of the faculty theory, 6, 7; popular inferences from, 40.
Cube root, 420–425; general principles involved in, 420; importance of, 425; method of finding, 424, 425.
Curiosity, teaching aided by, 179.
Curriculum, changes in, 42; democracy and, 43; educational abyss due to, 59, 60; elements of value in, 63–66; enrichment of modern, 42; graphs in, 418, 419; of easy tasks, 42, 44, 55, 59; practical purpose of, 180–188, 245, 331, 355; savings and investments in, 363–366; stocks and bonds in, 371, 372.

Darwin, Charles, 161, 162.
Dearborn, 10.
Decimals, accuracy of product of, 274–276; addition and subtraction of, 270, 271, 274–276; and common fractions, 266, 267; approximate, and approximate integral numbers, 268–270, 276; division of, 273, 274; early use of, 266; multiplication of, 271–273; reasons for using, 267, 268.
Democracy, constituents of a, 43; educating a, 43; educational requirements of the members of a, 45; equality in a, 43.
Denominate numbers, defined, 280, 281; fundamental operations on, 284, 285;

in measurement, 280–285; reduction of, 284; tables of, 281–283.
Dewey, John, 31, 91, 94. *See* McLellan and Dewey.
Discount, 329–335; bank, 389; computing of, 331, 332; defined, 329; occupations requiring a knowledge of, 334, 335; reasons for giving, 329–331; series, 332–334.
Distinction, personal, as a motivating force, 147, 148.
Division, defined and discussed, 305–308; learned along with the multiplication table, 211, 212; long, 240–243; of decimals, 273, 274; of mixed numbers, 260, 261; one-figure quotients and one-figure divisors in, 238, 239; problems solved by, 310, 311; short, 239, 240.
Drills, in fundamentals, 246, 247; on fundamental number combinations, 215–218; in percentage, 291.
Dunlap and Wells, 10.

Ebert and Meumann, 10.
Education, aims of, 106–108, 181, 287, 288; as a preparation for life, 44; doctrine of nontransfer of abilities developed by, 44; effectiveness developed through, 52, 53; increases intellectual differences, 148; early specialization in, 46, 47; judging the success of new methods in, 109–112; origin of project method in, 108; pleasure of activity as an aim in, 131, 132; results of, 37, 57; the 'practical' in, 44.
Educational corollaries, 4.
Educational credo, 53.
Effectiveness, constituents of, 52; defined, 14, 15; differences in, 50, 51; education for developing, 52, 53; equation of, 15; illustrations of personal, 49.
Efficiency, gained through interest, 158–160, 186–188.
Effort, doctrine of value of, in mental exercise, 42; essential in bodily exercise, 41; expended to make money, 46; failure results in abandonment of, 54; school curriculum and, 42, 43; skill in directing intellectual, 55.
Elements, of educational value in the course of study, 63–66.
Elements of the project method as determined by Stevenson, 94.
Empirical learning, 80, 81.
Enterprise, defined, 13, 14; discussion of, 39, 40; encouragement of, 66; equation of, 15.
Epstein, 9.
Exercise, doctrine of value of effort in mental, 42; effort essential in bodily, 41.
Experience, spread of form of, 18; transfer of material of, 19.
Experiments, 9–11, 24–40; general characteristics of, in laboratory training, 11; popular inferences from, 40.
Express money orders, 392.

Faculty theory, modern criticism of, 6, 7.
Failure, learning by, 62, 63.
Farm arithmetic, 441–450; cost of feeding farm animals, 442, 443; farm income statement, 448, 449; farm inventories, 444, 445; labor required in farming operations, 443, 444; marketing farm products, 445, 446; mensuration for the farm, 450; need for, 441, 442; nutritive ratio and balanced rations, 446–448; record of receipts and expenditures, 449, 450.
Fatigue, ability to overcome, 37–39; analogy of intellectual and physical, 35, 36; decreases ability, 36; defined, 37; industry and, 35.
Figure-blind people, 417, 418.
Figures, use of, in teaching fractions, 254.
Foreign money, 285, 286.
Formal discipline, 3–68; doctrine of, abrogated, 40, 58, 59; belief in, of people in general, 47, 48; in analytical form, 5; opposition to, 30; preparation for life according to, 44; principles *versus* memorizing rules, a problem in,

183–186; theory of, in simpler form, 3, 4.
Fractions, addition of, 249, 261; cancellation in, 252–253; common, 249–265; decimal, 266–277; division of, 258–261; early introduction of, 249; fundamental property of, 251, 252; in common use, 250, 251; learned along with the multiplication table, 211, 212; multiplication of, 257, 258; oral drill in, 261, 262; oral exercises in, 255; reducing, to common denominators, 254, 256; rules for solution of, 322; three problems in, 263, 264; subtraction of, 249, 256, 257; use of figures in teaching, 254.
Fraker. *See* Gilbert and Fraker.
Fundamental number combinations, drills on, 215, 216.
Fundamental operations on whole numbers, 221–247.
Fundamentals, drills in, 246, 247.

Gamble, 10.
Games, examples of bad, 143, 144; examples of good, 144–147; in the upper grades, 155; of rivalry, as motivating forces, 129, 141–149.
Geometrical relations, 178.
Gilbert and Fraker, 9.
Grammar grades, elements of a good course in arithmetic in, 172–174.
Graphic representations, 412–419.
Graphs, bar, 412, 413; curve, 414–416; for figure-blind people, 417, 418; importance of, in the curriculum, 418, 419; material for, 416, 417; uses of, 412, 417.
Group rivalry, 129, 141–149.
Grube, 193.

Handel, 74.
Heaviside, Oliver, 225.
Holloway, 205.
Home arithmetic, 433–440; cost of a servant, 435; cost of food, 435–437; cost of owning a car, 434, 435; cost of owning or furnishing a house, 435; home as a business institution, 433, 434; record of expenditures, 437–440.
Hughes, Charles E., 44, 60.

Ideals, general rather than specific quality of, 157; imitation of, 157; permanent, 166, 167.
Imitation, as a basis for interest, 130; as a motivating force, 157; of adults, 157; selective, 157.
Individual tests, method of giving, 150; records of, a stimulating force, 150, 151.
Industry, analogy of intellectual and physical, 35, 36; defined, 13, 14; development of, 65, 66; equation of, 15; fatigue and, 35; incentives to, 37, 38.
Initiative, discussion and example of, 60–63.
Insurance, 403–410; fire, 403–406; life, 406–410; property, 403.
Interest, derived from immediate present, 121; derived from remote considerations, 123; derived or developed, 135, 156; histrionic, 135; imitation as a basis for, 130; in human affairs, a motivating force, 157, 158, 186–188; in personal distinction and rivalry, 127; in things already known, 129; in what is new, 130; kinds of, 120; nature of, 120; results in efficiency, 158–160, 186–188; shown in muscular activity, 126; success as a factor in, 133; teaching aided by, 179.
Interest (as a subject in arithmetic), 344–355; bankers' method of computing, 349–351; basis for computing, 346, 347; comparison of methods for computing, 352–354; direct method of computing, 349; element of time in computing, 347–349; finding rate, principal, and time in problems on, 354, 355; local applications of, 355; on bonds, 378; paid by savings banks, 384, 385; prevalence of loans and, 344, 345; profit and, 345; reasons for paying, 346; safe rates of, 361–363; six

per cent method of computing, 351, 352; use of tables of, 352.
Interests, selection of, 136.
Investigations, by Holloway in addition combinations, 198; by Holloway in multiplication combinations, 202; by Jessup and Coffman in subtraction methods, 233, 234; by Montana teachers in addition combinations, 200–202; by Montana teachers in multiplication combinations, 202–204; by Phelps in addition combinations, 199; in spread of ability, 9–11; in transfer of training, 24–40.
Investments. *See* Savings and investments.

James, William, 9, 56, 57, 179, 180.
Jastrow and Cairnes, 9.
Jessup and Coffman, investigation in subtraction methods by, 233, 234.
Judd, C. H., 23, 25. *See* Schalckow and Judd.
Judd and Cowling, 10.

Klapper, Paul, 289, 318.
Kline, 10.

L. C. M., method for finding, 262, 263.
Language, accuracy of, in arithmetic, 181–183.
Learn, to learn, 62, 63; to do, 62, 63.
Learning, by failure, 62, 63; by success, 62, 63; common elements in processes of, 58, 59; empirical, 80, 81; to farm, 70–76; to keep house, 76, 77; methods of, 69–117; of adults, 77–80; relation of transfer to manner of, 23, 24; to build, 73–76.
Lennes and Jenkins, 246, 254.
Liddle, 10.
Life. *See* School and life.
Life motive, President Eliot's definition of, 39.
Loss. *See* Profit and loss.

Market price, discount, gains, 339–340.
McLellan, J. A., 31.
McLellan and Dewey, 186, 192, 193, 194, 254, 289.
Measurement, defined, 278; denominate numbers in, 280–285; development of process of, in counting, 192; inaccuracy of, 279, 280; mensuration and, 426; metric system in, 285, 287; units of, 178, 179.
Mensuration, 426–432; areas and volumes found by, 428–432; defined, 426; farm, 450; in lower grades, 428; measurement and, 426; practical importance of topics in, 431, 432; Pythagorean proposition in, 426, 427; use of similar triangles in, 426–428.
Method, project, 92–101; scientific, 83, 84; topical, 84–92.
Methods, judging the success of new, in education, 109–112; of experimental psychologists, 7; of learning and teaching, 69–117; value of far-reaching, in solution of problems, 311–313.
Meumann. *See* Ebert and Meumann.
Mixed numbers, division of, 260, 261; multiplication of, 257, 258; subtraction of, 256.
Money orders, 391, 392.
Moore, E. C., 44. *See* Angell and Moore.
Motion, first law of, by Newton, 81.
Motivated acts, 121–125, 160–162.
Motivating forces, combative instincts as, 128; curiosity, pleasure in success, instinct of workmanship, personal and group rivalry as lasting, 155; games of rivalry as, 128, 129, 141–149; group instincts and rivalry as, 128, 129, 141–147; love of work *versus*, 165; personal distinction and rivalry as, 128, 147, 148.
Motivation, artificial, 164; curiosity as, 134; defined, 125; derived interest as, 135; desire for excellence as, 133; drill on fundamental number combinations as, 215–218; histrionic interest as, 135; human significance of studies as, 186–188; in desire to construct things, 127; in early grades, 119–155; in grammar grades, 155–167; in play,

126; interest in things already known as, 129; interest in what is new as, 130; means of, 162–165; most effective elements in, 140; pleasure of activity as, 131; results of proper, 166, 167; rivalry as, 147–150; selective imitation as, 157; success as, 133; value of, 165–167.

Motives, of the adult, 160–162; selection of, 162–165; transient and permanent, 160–162.

Multiplicand, 237, 238.

Multiplication, basic ideas in, 235–237; by one-figure numbers, 234, 235; comparative difficulty of, 202–204; memorizing necessary in, 197; multiplier and multiplicand in, 237, 238; of decimals, 271–273; of fractions, 257, 258; of mixed numbers, 257–258; solution of problems in, 299–302.

Multiplication table and related number facts, 211, 212, 234, 235.

Multiplier, 237, 238.

"Natural setting," discussion of, 112–115.

N. E. A. *Proceedings*, 177.

Newton, Sir Isaac, 81.

Numbers and number combinations, 191–220; addition by endings, 221–224; analysis of, in early grades, 193, 194; carrying in addition, 224–226; child's list of difficult, 204, 205; column additions, 224; concrete basis for, 194–196; drill on fundamental, 215–218; early use of large, 226, 227; mastering, separately, 213, 214; order of introducing, 214, 215; operations on concrete and abstract, 314–318; practical applications of, 218; practical purpose of concrete work in, 196, 197; purpose of memorizing, 197. *See* Whole numbers, Combinations.

Objects, to be achieved, 171–189.

Paying bills, at a distance, 391–394; by bankers' drafts, 393; by personal checks, 392, 393; by postal orders and express orders, 392; reasons for, 391; service of banks in, 393, 394; why distant payments are not made in cash, 391, 392.

Percentage, 289–293; base, rate, and, 292; chief difficulty in teaching, 292, 293; drill on, 291; in discount, 331, 332; relation of, to decimals and common fractions, 289, 290; uses of, 291, 292.

Personal checks, 392, 393.

Phelps, C. L., 199.

Physics, topics from, 178, 179.

Play, drill on fundamental number combinations regarded as, 217, 218; examples of bad games, 143; examples of good games, 144; games of rivalry, 141; motivation in, 126.

Pleasure of activity, as a motivating force, an aim of education, 131, 132; dependent upon one's present adaptation for it, 152, 153.

Popular inferences, from modern criticism and experimentation, 40.

Postal money orders, 392.

Problems, addition-subtraction group of, 302–304; checking results in solutions of, 326, 327; difficulties in the simplest one-step, 295–297; division, 310; forms of solutions of, 323–325; fundamental relations of, 311–313; general suggestions on solving, 327; indicated solutions in, 321–323; multiplication-division group of, 305–310; organizing the solutions of, 294, 295, 297–301; solving one-step, by analysis, 318–321; use of "crutches" in solution of, 326; without numbers, 313, 314.

Processes, practical needs of fundamental, 245; rationalizing the, 243–245.

Proctor, 323.

Profit and loss, 336–340; defined, 336, 337; interest and profit, 345; market price, discounts, gains, 339, 340; occupations requiring a knowledge of, 340; computed on buying price, 337, 338;

computed on selling price, 338, 339; why there must be profits, 337.
Project method, discussion of, 92; elements of, as determined by Stevenson, 94; examples of, 93–96; origin of, in education, 108.
Projects, examples of, 97–101; possible use of, in arithmetic, 115–117, 187, 188.
Psychology, influence of, on life and thought, 31.
Pupils' record cards, 151.
Pythagorean proposition in mensuration, 426, 427.

Rall, 11.
Ratio, of numbers, 193; nutritive, 446–448.
Reasoning, differences in, 31.
Record cards, pupils', 151.
Relation of facts, 223, 224.
Rivalry, as a motivating force, 163, 188; ethical quality of, 147–150; games of, 129, 141–149; lessons learned in group, 149, 150; personal *versus* group, 148–150.
Ruediger, W. C., 24.
Ruger, H. A., 10, 26.
Rugg, H. O., 7, 27, 28, 29, 30, 32, 33.

Savings and investments, 357–370; examples on savings, 366–370; kinds of investments to avoid, 359, 360; meaning of investments, 358, 359; meaning of savings, 357, 358; place in curriculum for, 363–366; rates of income on investments, 360, 361; safe rates of interest, 361–363.
Schalckow and Judd, 10.
School and life, differences between, 82; problem of, 69; relation of, 69.
Schwab, Charles M., 149.
Scientific method, discussion of, 83, 84.
Self-direction, discussion and example of, 60–63.
Sleight, 11.
Smith, David Eugene, 174, 196, 289, 303.
Solutions of problems, 294–327; checking results in, 326, 327; fallibility of, based solely on common elements, 55–57; forms of, 323–325; general suggestion on, 327; in addition, 297, 298, 300–302; in division, 310, 311; indicated, 321–323; in multiplication, 299, 302; in subtraction, 299, 301, 302; of one-step type by analysis, 318–321; organizing the, 294, 295, 297–301; rules for, in fractions, 322; value of far-reaching methods in, 311–313; use of "crutches" in, 326.
Specialization, difficulties of early, 46, 47; education for, 46, 47; early, 45.
Spencer, Herbert, 48, 50.
Spread, of ability by laboratory training, 8; amount and character of, 12, 13; of "form of experience," 18.
Square root, 420–425; general principles involved in, 420; importance of, 425; methods of finding, 421–424.
Squire, Carrie R., 24.
Starch, Daniel, 7, 207, 213, 246, 247; investigation by, 34.
Stevenson, John A., 93, 94, 95, 96, 97, 98, 99, 100, 102, 108.
Stimson, R. W., 98, 99.
Stocks and bonds, 371–382; brokerage, 379; common, preferred, and cumulative preferred stock, 376; "debenture bonds," 376; dividends on stocks, 378; essential simplicity of, 381, 382; importance of, in curriculum, 371, 372; industrial bonds, 375; interest on bonds, 378; municipal and government bonds, 375; organization of a stock company, 372, 373; par and market values of, 377, 378; problems included under, 382; rate of income from investments in, 380, 381; registered and coupon bonds, 376, 377; contrasted as to security, 378, 379.
Subtraction, Austrian method of, 209, 210, 232–234, 304; comparison of methods of, 232–234; defined, 306; methods of, 209, 210, 227–231; methods of teaching, 207–210; of decimals,

271, 275, 276; relations, 177; solution of problems in, 299, 301, 302.
Success, as a basis of self-esteem, 133; as an element in interest, 133; confidence produced by, 53–55; defined 14, 15; equation of, 15; learning by, 62, 63.
Supervised study, in high schools and colleges, 60, 61.
Suzzallo, Henry, 165, 196, 225, 226, 331.
Swift, E. J., 80, 81.

Taking-away-borrowing method of subtraction, 227–230, 232–234.
Taking-away-carrying method of subtraction, 230–234.
Taxes, 395–401; average, per capita, 396, 397; expenditure of, 395; import duties, 399, 400; indirect, 401; reasons for increases in, 395, 396; sources of revenue, 397; spreading the tax, 398, 399; tax rate, 397, 398; tax table, 399.
Teaching, methods of learning and, 69–117.
Textbooks, as permanent reference books, 287, 288.
Thorndike, E. L., 36, 37, 38, 39, 42, 49, 54, 80, 91, 92, 132, 138, 160, 165, 176, 186, 221, 237, 423.
Thorndike and Woodworth, 9.
Topical method, criticism of, 86, 87; discussion of, 84–86; rejoinder to criticism of, 88–92.
Training, ability developed by, 19; in neatness, 24, 25; results of, in school course, 27; transfer of effects of, 19.
Transfer, of common elements in abilities, 20; of material of experience, 19; of results of training, 31; reasons for, 32; relation of, to manner of learning, 23, 24; value of, 34.
Transfer of training, amount and character of, 11, 12; distinction between spread of effects and, 17, 26, 27; practical demonstrations of, 64.

Urbantschitsch, 9.

Variation, in mental aptitudes, 43.
Vogt, 9.
Volumes, computed by mensuration, 428–432.

Wallin, 11.
Walsh, J. H., 175, 176, 289.
Webb, 11.
Weights and measures, units of, 178, 179.
Wells. *See* Dunlap and Wells.
Whipple, G. M., 33, 34.
Whole numbers, fundamental operations on, 221–248. *See* Numbers and number combinations.
Winch, 10.
Woodworth. *See* Thorndike and Woodworth.

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